Art. I.—*The "Three Sections," the "Tangencies," and a "Loci Problem" of Apollonius, and Porismatic Developments.* By Martin Gardiner, C. E. (formerly Science Scholar, Queen's College, Galway).

[Read before the Royal Society, June 4, 1860.]

PRELIMINARY OBSERVATIONS.

In the Transactions for 1859 I promised solutions to the celebrated problems of the Greek and French schools, and the present paper is the first instalment towards the fulfilment of that promise.

I commence with the problems of Apollonius, known as his "Three Sections," and "Tangencies," and the principal problem of his treatise on Loci; but I propose also the continuation of the development of interesting "Porisms."

The problems of the Three Sections are famous from the number of geometers who have assayed their solutions.

Willebrord Snel (the first person who measured the length of an arc of the meridian by means of a geodetic survey), who was born at Oudewater, in Holland, in the year 1590, was the first geometer of eminence to restore the Section of Ratio. His solution was published at Leyden, in 1608. Early in the eighteenth century, Dr. Halley discovered an Arabic manuscript in the Bodleian Library containing distinct investigations to the numerous subdivisions of the Section of Ratio, a Latin edition of which he published at Oxford in the year 1706; but there is no evidence as to whether this relic is a transcript from the original of Apollonius, or merely a string of solutions to its various cases by some other geometer; it covers 138 pages. Since then the principal solution is that by Reuben Burrow, which was published about the year 1780 in his "Apollonius." An application of the problem may be seen in David Gregory's Astronomy.

The Section of Space received an original solution from Dr. Halley, which is similar to that given in Leslie's Geometrical Analysis. Other solutions may be found scattered through mathematical periodicals; but as they are all similar and incomplete, they deserve no particular notice.

Indeed, an unaccountable neglect has been shown to this problem by the geometers who attempted the other "Sections;" and this is the more strange as the Section of Space is by far the most useful of the three.

The Determinate Section was solved by Willebrord Snel, and since then by Dr. Robert Simson, William Wales, and Petro Giannini. Snel's and Wales' solutions were republished at London in 1772, by the Rev. John Lawson, and Giannini's at Parma in 1773. Dr. Simson's solution was published in his Opera Reliqua in 1776, at the private expense of Earl Stanhope, and covers over 150 pages.

However, though the lost writings of Apollonius occupied the attention of Newton, Halley, Simson, Burrow, Huygens, D'Omerique, Lalouerc, and a host of other distinguished geometers, it is a most remarkable fact that none of them perceived the *liaison* of "The Three Sections." Indeed, it was only through the instrumentality of the *Homographic Theory,* as systematised by M. Chasles, Professor of Geometry to the Faculty of Sciences of Paris, that this intimate connection was exposed, and analogous solutions for the first time given. Chasles' solutions—extracted from his correspondence with the late Professor Davies, of the Royal Military Academy—dated 1848, were published in the third volume of the *Mathematician,* and again in his recent work entitled Traité de Géométrie Supérieure.

These latter solutions are more in detail than those in the Mathematician, and the following accompanying observations of the author, who has been justly styled the Newton of Geometry, are worthy of special attention. He says :—

"Amongst the numerous questions to which the homographic theory can be most easily applied, are those which formed the subject of the three works of Apollonius, entitled the Section of Ratio, the Section of Space, and the Determinate Section. Each of these questions exacted a great number of propositions. Pappus relates that there were 181 in the Section of Ratio, 124 in the Section of Space, and 83 in the Determinate Section. These arose from the fact that the solution to the general question was never given directly, as the ancient geometers proceeded to first establish the most simple cases, and then went step by step to the more general, so that the solution of each case always depended on those which preceded. Moreover, each problem gave rise to as many different questions as there were varieties in the different relative parts of the figure.

In the two last centuries these problems have occupied the attention of many eminent geometers, who endeavoured to restore the works of Apollonius; but, although they tried to reduce the solution of each to as few propositions as possible, it is yet the same long and tortuous method they have all followed. For instance, J. Leslie gives four propositions to the Section of Ratio, six to the Section of Space, and eight to the Determinate Section, whilst, by my method, one solution suffices for the three questions, considered in their most general forms."

Now, I have already recorded my opinion concerning the peculiar method of investigation of the ancient geometers and their modern imitators, namely, that it is attributable to the want of precision and generality in the indicated operations, and involved theorems; but I will further observe, in this place, that the homographic theory must receive some developments in *limits* to the constants of the equations, implicating the *double points* of divisions on the same straight line, before it becomes thoroughly effective in its applications. And from the absence of such developments, Chasles' solutions are necessarily defective.

Take for instance his solution to the Section of Ratio, which is as follows*:—

"Draw AE parallel to NN, to cut MM in E; draw AG parallel to MM to cut NN in G; find I in MM such that PI : RG :: m : n; bisect IE in O; in NN find H such that PO : RH :: m : n; draw HA to cut MM in F; from O as centre and radius = $(\mathrm{OF}\cdot\mathrm{OE})^{\frac{1}{2}}$ describe a circle; through either point C in which this circle cuts MM, draw CA to cut NN in D: then will CAD be an answerable line." And his only remarks in respect to the limits of the problem are—"And if the segments OF and OE be not on the same side of O, the two solutions will be imaginary."

Here it is evident that when the given straight lines MM, NN, are parallel, the method is not intelligibly applicable. And it is but right to observe that this is the only case in which the principal construction given in Leslie's Geometrical Analysis (introducing the improvement of indicating opposite directions by opposite signs) cannot be applied. However, the general method of finding the *double points* of homographic divisions which is given in the Géométrie Supérieure, would, if introduced, overcome this imperfection. But there is a much more serious defect which cannot be rectified by the "theory," such as it now exists, namely, the non-establish-

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* See the enunciation I give to this problem.

ment of the precise limiting values for the ratio $\frac{m}{n}$. Surely, it is not evident that there are two limiting positions for F, such as f and f', and that according as the ratio $\frac{m}{n}$ lies outside the limits $\frac{\text{P}f}{\text{R}h}$ and $\frac{\text{P}f'}{\text{R}h'}$ (h and h' being the points in which fA and f'A cut NN), or is equal to one of them, or is comprehended between them, so will the corresponding points C be real and distinct, real and coincident, or imaginary.

And similar remarks apply to his solutions to the Section of Space and to the Determinate Section; for the homographic theory will not establish the *limits,* nor even hint as to their nature or number.

My solutions are equally general with those given by Chasles, and—as will be seen in the Generating Problem—one wording applies to the three questions in their most general forms. Besides, they possess the distinguishing characteristic of being intelligibly applicable to all the particular cases; and the simple considerations, by means of which the limits are established, will be found applicable to the determinations of limits in numerous other important questions.

The next in order of the works of Apollonius, after the Determinate Section, was the "Tangencies."

The enunciation of the problem, and some of the "Lemmas" used in its solution, which were preserved in the Mathematical Collection of Pappus, enabled Dr. Robert Simson, of Glasgow, to reproduce one case (that of two circles and a point) though not under its original form,—as may be seen in the Appendix to his Opera Reliqua; but a more elegant solution to the same was previously given by Vieta, in his Apollonius Gallus. And since Dr. Simson's, an entirely different solution has been given by Monsieur Auguste Cauchy, in the "Correspondence de l'Ecole Polytechnique."

However, neither Simson, Vieta, nor Cauchy succeeded in giving a direct solution to the general question.

Newton virtually solved the general question in his Principiæ, where it entered into some astronomical determinations; and, indeed, it is the only direct geometrical solution by a British geometer which applies to the various cases, when we suppose the circles to have any value from zero to infinity.

But the most complete and elegant solution hitherto given to the "Tangencies," is that of M. Gergonne, in the *Annales de Mathematique,* which (according to M. Chasles) is an improvement on a solution by M. Gaultier, in the *Journal de l'Ecole Polytechnique.*

In this paper I give ten direct geometrical solutions to the general question.

The first of these is, I consider, the simplest ever given Its applications to the case in which two of the circles are finite, and the other circle infinitely small, is an improvement on Vieta's solution; and to the case where two of the given circles are infinitely small, and the third finite, it is similar to what is given by Brianchon* in the Journal de l'Ecole Polytechnique.

The second solution is also applicable to all the cases of the problem; and the idea of the auxiliary circle can be applied in other questions, so as to render the solutions more general.

The application of the third solution to the case, in which two of the given circles are finite, and the third infinitely small, leads to M. Cauchy's method for this case, &c.

The other solutions are applicable to all the leading cases of the problem, but fail to indicate graphical constructions for some of the minor ones, owing to the peculiarities inherent in the involved theorems, or in the methods of contemplating or expounding them. The tenth is most probably a reproduction of Apollonius' solution.

The "Loci Problem," which I have undertaken, is in a more general form than was accorded to it by Apollonius. It comprehends almost the entire substance of the Second Book, as restored by Dr. Simson.

The solution is direct and general; besides, it shows that when the *ratio* is unrestricted in sign, the locus is not (as usually intimated) *a circle,* but two real circles, a real circle and a point, or a real circle and an imaginary one, according to relative states of the data.

Particular cases only of this problem were solved by Dr. Simson, all of which have been republished in Leslie's Geometrical Analysis. His methods are inapplicable to the general question, as they depend on the reality of a point in the straight line passing through the given points, which may become imaginary, even when the locus is real.

A method of constructing the locus, having many points approaching to mine, is given in the Geometry of the Library of Useful Knowledge; but there, too, the process depending on points which may be imaginary when the locus

* Professor Davies has erroneously confounded Brianchon's with Pappus' solution.—(See vol. 3, page 227, Mathematician.)

is real, is applicable only in particular states of the data. In the Notes will be found a genuine *ancient* porism, from which the problem originated.

I might also mention that Francis Van Schooten, Professor of Mathematics at Leyden, published a restoration of some of the particular cases of this problem, in 1657; and that a like task was performed in an algebraic form by Fermat, Councillor to the Parliament of Toulouse, in his Opera Varia Mathematica, published in the year 1679.

It is scarcely necessary to remark, that in the present improved state of Algebraic Geometry, it would be an easy matter to solve the general case of this problem; but, to arrive at a construction of the Locus, such as I give, would be impossible without introducing other geometrical considerations than those to be found in ordinary Algebraic Elements; besides, the complete discussion would present difficulties which none but experienced analysts could overcome.

The "Porisms" in the present paper, with those in the *Transactions* for 1859, belong to the most numerous and useful system in the whole range of elementary theorems. Some few of them—as is evident from Professor Davies' contributions to the Mathematician—have been already noticed by Mr. Mark Noble, and by Professors Playfair and Wallace; but their number is so few, that when they occur I will make no scruple of reproducing them amongst the classes to which they belong. I have already given proof of their efficiency in the solution of difficult problems.

They are most probably but restitutions of a part of the lost treatise of Euclid, known as his Second Elements—composed when his geometrical knowledge was fully matured, and which, there is strong reason to suspect, contained all the principles developed in the elementary writings of Gergonne, Poncelet and Chasles.

Having said thus much relating to the substance of my paper, I think it right, before closing these preliminary remarks, to explain the nature of the improved ideas and theorems on which the *spirit* of my investigations is mainly dependent.

To do this, I may at once state that all the great masters of Logic have observed that there are two points which must be rigorously attended to in correct systems of reasoning.

First:—That the propositions employed as premises are unambiguous, and correctly understood.

Second:—That the steps (the auxiliary operations and theorems) by means of which the conclusion is drawn from those premises, are true, unambiguous, and correctly understood.

This being borne in mind, it is evident that if from a given point in a given indefinite straight line, we were told to cut off a part equal in length to a given finite straight line, we should naturally ask in which direction from the point we are to take the part; as the problem would be ambiguous if either direction should not be answerable to the end in view. And if the solution of some other problem depended on this operation (as just defined), and that one only of the parts which can be cut off is applicable, then it is evident there would exist an ambiguity in the solution.

The method of indicating opposite directions on the same straight line in distinct terms—such as positive and negative directions, or right and left directions, obviates this difficulty; but though long since adopted in Trigonometry and Algebraic Geometry, it is only in the modern French pure geometry, that it has been consistently introduced.

Again, if in any general investigation or construction it were necessary to draw a straight line through a particular point, making an angle of a given magnitude with another straight line, then, as two such lines can be drawn through the point, and that but one of them may be answerable, it is clear there should be a precise method of indicating each of these lines.

Further, if on any straight line, for instance, a particularised one of the last two, it were necessary to cut off a segment from a point therein whose length should have some peculiar relation to other magnitudes and positions, and that but one segment from the point would be answerable; then, too, it is obvious we should have a method of particularising directions in one straight line in respect to the directions in others.

Yet it is only in my previous papers such methods are either advocated or applied.

And without those improvements in the manner of indicating angles, it is not only the elementary geometry of the straight line and circle that suffers, but also the conic sections and higher departments; for there, too, geometers have failed to expose the general truths comprehended in the theory. One instance of this is supplied by the following well-known theorem:—"When the base AB of a triangle is given in position and magnitude, and that the difference of the angles CAB, CBA at the base is constant, then will the locus of the vertex C be a hyperbola."

For, as the locus of the vertex is not a hyperbola under these conditions, but part of one, it follows that, in all inquiries in which the theorem is used, the results must be defective in generality.

The complete theorem which should replace it is as follows:—

If *aa* and *bb* are fixed straight lines through fixed points A and B, should *cc* and *dd* be any other pair of straight lines through A and B, making the angle *cc* right to *aa* equal to the angle *bb* right to *dd*, then will the locus of the intersection of *cc* and *dd* be a rectangular hyperbola through the fixed points A and B.

Indeed, a due consideration of the requirements of a complete logic, or of the laws of nature, will show that the improvements are necessary to the explicit enunciations of implied operations, and confer precision and generality on most important theorems; and are therefore, in so far as these are understood, a correct step in the advancement of pure science.

Finally, I think it right to remark that, in indicating opposite formations of magnitudes, I have purposely avoided the terms *positive* and *negative*, as so many meanings are given to these words by metaphysicians and others. Besides, the introduction of *right* and *left* renders the language more elegant, and often affords important advantages in allowing us to decide, according to circumstances, whether right or left should be indicated by plus or minus.

MARTIN GARDINER, C.E.

DEFINITIONS.

1. If a straight line, which we may conceive produced to infinity in its primitive directions, be supposed to become rigid, and one point of it to be permanently fixed, the rigid line being otherwise capable of movement in any plane in which it may lie, then it is evident that there are but TWO ways of revolving the line in this plane; one being by means of a "*right*" rotation, and the same as that in which the hands of a watch move if the dial-plate be towards us, in the plane; and the other being by means of the contrary, or "*left*" rotation.

2. If AA and BB be two straight lines, and I their point of intersection, then the "*angle IA right to B*," means the angle formed at I by a rigid line having I as a fixed pivot, and

revolving from a position in AA by a right rotation until its first arrival into the position BB, the revolving line being supposed produced indefinitely on both sides of the pivot. And a similar meaning applies to the term "*angle IA left to B.*"

3. If AA and BB be two straight lines, and I their point of intersection, then "*angle IA right round to IB*" means the angle formed at I by a straight line having one of its extremities in this point, revolved by right rotation from the actual direction IA until it arrives in the actual direction IB. And a similar meaning applies to the term "*angle IA left round to IB.*"

4. If AA and BB be two straight lines, and I their point of intersection, the angle "*right AB*" means the angle IA right round to IB, and the angle "*left AB*" means the angle IA left round to IB.

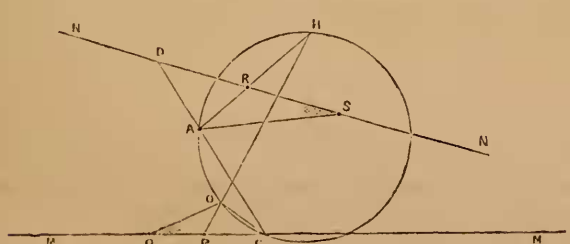
5. The angle (AB) means either one or the other of these last two, indifferently.

NOTE.

It is necessary to restrict the meaning of the term "angle (AB)," given in Chasles' Géométrie Supérieure, to that which has been just defined; for otherwise, his enunciated properties of the homographic pencils will not hold good as to sign.

See my paper entitled "Improvements in Fundamental Ideas and Elementary Theorems of Geometry," in the *Transactions* for 1859.

SECTION OF RATIO.



Given the points P *and* R *in the given straight lines* MM *and* NN; *through a given point* A *to draw a straight line* CAD *to cut the given lines in* C *and* D, *so that the segments*

PC *and* RD *shall be to each other in the given ratio of* m *to* n, *(* $\frac{m}{n}$ *being of given magnitude, and of known sign in respect to directions on* MM *and* NN*).*

ANALYSIS.

Suppose on MM and NN we take PQ and RS, so that PQ : RS :: PC : RD :: m : n, and that we draw PO and QO making the angles PO right to Q, and QO right to P respectively equal to the angles RA right to S, and SA right to R. Then it is evident that the triangle POC is similar to RAD, and that the angle OC right to P is equal to the angle AD right to R. Hence, H being the point of intersection of PO and RA, it follows that a circle can pass through AOC and H; but A, O, and H are known points: therefore the point C, in which the circle AHO cuts MM, is known, and therefore also the line CAD.

COMPOSITION.

On MM and NN take segments PQ and RS, having to each other the given ratio of m : n; draw PO and QO, making the angles PO right to M, and QO right to M equal respectively to the angles RA right to N, and SA right to N; through A, O, and the intersection H of PO and AR, describe a circle; through either point C, in which this circle AHO cuts MM, draw CA to cut NN in D: then will CAD be an answerable line.

For draw OC. The angle AH or AR right to C or D is equal angle OH or OP right to C, and therefore since the angle PO right to Q or C is equal angle RA right to S or D, it is evident that the triangles POC and RAD are similar, and that PC : RD :: PO : RA :: PQ : RS :: m : n.

DISCUSSION.

It is evident that when $\frac{m}{n}$ is restricted as to sign, there is but one point O, one circle OAH, and two answerable points C (real or unreal).

If $\frac{m}{n}$ be unrestricted in sign, then, obviously, there are two points O, two corresponding circles AOH, and, therefore, four answerable points C. Moreover, as the points O must be on opposite sides of MM, it is evident two of these points C must be always real.

Limiting Values for the Ratio $\frac{m}{n}$.

When A and H are on opposite sides of MM, the corresponding points C are always real; but when A and H are on the same side of MM, the reality of the points C is dependent on the position of O, or, which amounts to the same thing, on the value of $\frac{m}{n}$. Again, since PO : RA : : PQ : RS, it is evident that if O′ and O′ be the points in which the two circles through, A and H touching MM again cut PH, then will $\frac{PO'}{RA}$ and $\frac{PO'}{RA}$ be the limiting values of $\frac{m}{n}$. Moreover, it is evident that, according as any value of $\frac{m}{n}$ is comprehended between these limits, or equal to one of them, or not comprehended between them, so will the corresponding points C be imaginary, or real and coincident, or real and distinct.

Porismatic Relations of the Data.

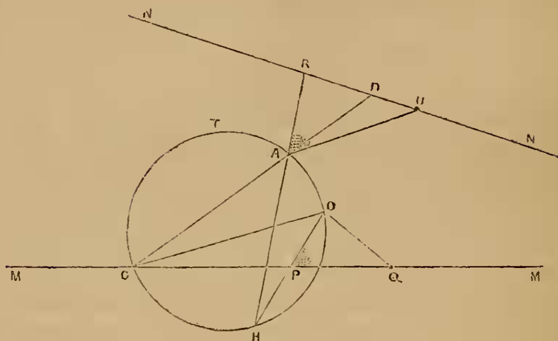
It is evident the problem is indeterminate only when the circle AOH is indeterminate. When O coincides with A, and that PO cuts AH, then H also coincides with A, and the circle AOH is infinitely small; but when O coincides with A, and that PO and RA form one straight line, then it is obvious that any circle touching this straight line in A is an answerable circle AHO: therefore in this case there are innumerable answerable points C and lines CAD. The problem under these last conditions (viz., when we have MM parallel to NN, and PRA a straight line, and the ratio $\frac{m}{n}$ equal to $\frac{PA}{RA}$) is said to be "porismatic"—any straight line CAD through A being an answerable line.

Remarks.

1. When MM and NN are parallels, it is evident PO and RA are parallels and that H is at infinity (when O and A are not coincident), and the circle AOH infinitely great. In this case the straight line AO, lying in the infinite circumference, will give one point C in its intersection with MM: the other point C is evidently at infinity on MM.

2. In all cases QO and SA intersect in the circumference of the circle AOH.

THE SECTION OF SPACE.



Given the points Q *and* U *in the given straight lines* MM *and* NN*; through a given point* A *to draw a straight line* CAD *cutting* MM *and* NN *in* C *and* D, *so that* QC.UD *shall be equal to* m.n *(where* m.n *is a given magnitude of known sign in respect to the directions on* MM *and* NN*).*

ANALYSIS.

Suppose that in the given lines we take the segments QP and UR such that QP.UR = QC.UD = *m.n*, and that we draw PO and QO making the angles PO right to M, and QO right to M, respectively equal to AR right to U, and UR right to A. Then the similar triangles POQ, ARU, give QO.UA = QP.UR, and therefore QO.UA = QC.UD; and ∴ as the angle QC right to O is equal the angle UA right to D, the triangles COQ and ADU are similar, and the angle CO right to Q is equal AD right to U; but PO right to Q is equal AR right to U; therefore it follows that angle AR right to D or C is equal OP right to C. Hence, if H be the point in which RA and PO intersect, a circle can pass through AOH and C: but the points AOH are known; therefore the point C in which the circle AOH cuts MM is known, and hence CAD.

COMPOSITION.

In the given lines MM and NN take any two segments QP and UR, such that QP.UR = *m.n*; draw PO and QO, making the angles PO right to Q, and QO right to P equal respectively to AR right to U, and UR right to A; through A, O, and the intersection H of RA and PO, describe a circle; through either point C in which this circle cuts MM draw CA to cut NN in D: then will CAD be an answerable line.

For draw CO.

The angle OH or OP right to C is = AH or AR right to C or D; and the angle PO right to Q = AR right to U; therefore the angle CO right to Q is equal AD right to U; hence the triangles ADU, COQ are similar, and QC.UD = QO.UA. But the similar triangles PQO, AUR give QO.UA = QP.UR; therefore QC.UD = QP.UR = *m.n*.

DISCUSSION.

When *m.n* is restricted in sign (as in the enunciation), there is evidently but one answerable point O, and therefore but one circle AOH, and two points C, real or unreal, according as the circle AOH cuts MM in real or imaginary points,

If *m.n* be unrestricted as to sign, then there are evidently two answerable points O, and therefore two circles AOH, and four points C, and lines CAD: moreover, it is evident that the points O are on opposite sides of MM, and therefore that two of the points C must be always real.

Limiting Values for m.n.

It is evident the points C can be imaginary only when A, O, and H are on the same side of MM. We know one point A in the circle AOHC, but, in order to arrive in a simple manner at the limiting positions for the circle AOH, it would be well if we could find another point in the circumference. We can find such a point. For, if T be the point in which QO again cuts the circle, the angle AH right to T = OH right to T, and is therefore = RA or RH right to N; and hence AT is parallel to NN, and the point T in which it cuts QO is known.

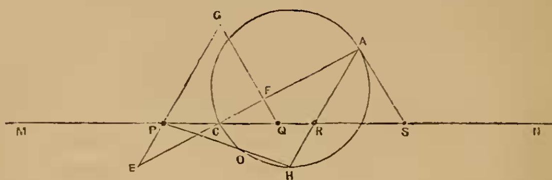
Now it is evident that by putting O′ and O′ for the points in which the circles through A and T, touching MM, cut QO, then will UA.QO′ and UA.QO′ be the required limits. Moreover, it is evident that according as any value of *m.n* is comprehended between these limits, or equal to one of them, or not comprehended between them, so will the corresponding points C be imaginary, or real and coincident, or real and distinct.

Porismatic Relations of the Data.

It is evident the problem becomes indeterminate only when the circle AOH becomes indeterminate. Now the circle

AOH evidently becomes indeterminate when O and A coincide, and that QOAT is parallel to NN. In this state of the data all points in MM are answerable points C, and the problem is said to be "porismatic." It is evident that UA is parallel to MM when QOA is to NN, and that *m.n* (in this porismatic case) is equal QA.UA.

THE DETERMINATE SECTION.



Given two pair of points P,S, *and* Q,R, *in a straight line* MN; *to find a point* C *in the line such that* PC . SC : QC . RC : l : k . *(in which the sign of* $\frac{l}{k}$ *is known, as well as its magnitude).*

ANALYSIS.

Suppose we assume a point A, and that we draw PG and QG making the angles PG right to Q, and QG right to P respectively equal to RA right to S, and SA right to R; and that E and F are the points in which AC cuts QG and PG.

The triangles CPE, CQF, are similar to CRA, CSA, and we evidently have PE . SA : QF . RA : : PC . SC : QC . RC : : l : k; and therefore PE has to QF the known ratio of RA.l to SA.k. Hence (see Porism 4 in *Transactions* for 1859), the circle EFG passes through a known point O in the circumference of circle PGQ, which is such that PO : QO : : PE : QF.

Again, the angle EF or EC right to O = GF or GQ right to O = PQ or PC right to O; therefore a circle can pass through EPC and O; and hence, as AR is parallel to PE, if H be the point of intersection of PO and AR, it follows that a circle can pass through OHA and C; but O, H, and A are known points; therefore the circle OHA is known, and also the point C in which it cuts MN.

COMPOSITION.

Assume a point A (not in the given line); draw PG and QG making the angles PG and QG right to M equal respectively to the angles RA and SA right to N; describe the circle PGQ, and in it find O (on the same or opposite sides of PQ with G according as $\frac{PG}{QG}$ and $\frac{l.RA}{k.SA}$ have like or unlike signs) such that PO : QO : : l.RA : k.SA; draw OP to cut AR in H; describe the circle OAH: either point C in which it cuts MN is an answerable point.

Let E and F be the points in which GP and GQ cut AC.

It is evident PE is parallel RA, and QF to SA, and therefore that PE.SA : QF.RA : : PC.SC : QC.RC. Again, the angle EP right to C being equal AR or AH right to C, it is equal OH or OP right to C; therefore a circle can pass through OCPE, and the angle EC right to O = PC or PQ right to O = GQ right to O; hence, a circle can pass through GFOE, and therefore (see Porism 4, *Transactions* for 1859), PE : QF : : PO : QO : : l.RA : k.SA, and therefore PE.SA : QF.RA : : l : k, and consequently PC.SC : QC.RC : : l : k.

DISCUSSION.

It is evident the point of intersection I of OQ and SA is in the circumference OHA (for the angle IQ or IO right to A = QO right to G = PO right to G, and ∴ = HO right to A).

When $\frac{l}{k}$ is (as is supposed in the enunciation) confined to a particular sign, there is but one point O, one circle AHO, and therefore two (and but two) answerable points P—both real or both imaginary.

But if $\frac{l}{k}$ were unrestricted in sign, it is evident there would be two points O, and therefore two circles OHA, and four points C. Moreover, since the points O must be on different sides of MN, two of these points C must be always real.

Limiting Values for the Ratio $\frac{l}{k}$.

When the segments PS and QR lie partly on each other, the points H and I lie on opposite sides of MN, and therefore the corresponding point C must be always real.

When one of the segments PS and QR lies entirely on the other, it is evident the points A and G are on the same side of MN, and therefore it is only when the ratio $\frac{l}{k}$ is positive, that O, H and A can be on the same side of MN; in other words, the points C are real for all real negative values of $\frac{l}{k}$; but for positive values of $\frac{l}{k}$ the points C are real only when the circle OHA cuts MN in real points. Hence, it is necessary to define the positive limiting values of $\frac{l}{k}$ so as to be enabled to know, *a priori*, when the corresponding points C are real, &c. This is done by drawing the two straight lines PO′H′ to cut the circle PGQ and line AR in O′ and H′ such that the two circles AO′H′ shall touch MN (these circles can be easily described, since the point X, in which AG cuts the circle PGQ, is common to all circles AHO. For the angle XG right to O = PG right to O = HA right to O = XA right to O); for then the limits are $\frac{RA.PO'}{SA.QO'}$ and $\frac{RA.PO'}{SA.QO'}$; and these limits are evidently such that according as any value of $\frac{l}{k}$ lies outside them, or is equal to one of them, or is comprehended between them, so will the corresponding points C be real and distinct, real and coincident, or imaginary.

When the segments PS and QR have no part in common, it is evident A and G are on opposite sides of MN, and that it is only when the given ratio is negative that the points OHIA can be on the same side of MN, and ∴ only that the points C can be imaginary. Hence, in this case, it is necessary to define the negative limiting values of $\frac{m}{n}$. These limiting values $\frac{RA.Po'}{SA.Qo'}$, $\frac{RA.Po'}{SA.Qo'}$, are obviously found in the same manner as in last case, and like remarks as to their nature apply.

When Q coincides with S, it is evident the points C are always real, and that one of them is coincident with QS.

When P coincides with R, it is evident the points C are real, and one of them in PR.

When R and S are coincident, and P and Q are distinct, then AR and AS are coincident, and G is at infinity, and ∴ O is in the straight line PQ; moreover, H coincides with RS; ∴ the points C are real, and one of them coincident with R.

When P and Q are coincident, and R and S distinct; then G coincides with P and O; but although the triangle POQ is infinitely small, it is known in species, and therefore POH is known in position, and hence the circle OHA. The points C are real, and one of them coincident with PQ.

Porismatic Relations of the Data.

If R and S be coincident, and that we have P and Q also coincident, then as we may have conceived PQ and RS to have had any peculiar relation, such as a constant ratio, &c., during their diminution, it is evident we may suppose G anywhere whatever in the line PQG. And it is clear that for all values of $\frac{l}{k}$ other than unity, the point O must coincide with PQ, and one point C be coincident with PQO, and the other point C with RSH. But for the value of $\frac{l}{k}$ = unity, the point O may be anywhere in the circle GQP (which touches MM in PQ), and ∴ a point C may be anywhere in MM—the state of the data being "porismatic." And it is evident that when P coincides with R and S with Q, and that $\frac{l}{k}$ = unity, then, also, will the problem be "porismatic."

Peculiar Case.

If in MN we suppose SR and RU equals respectively to l and k; then we have PC.SC : QC.RC : : SR : RU.

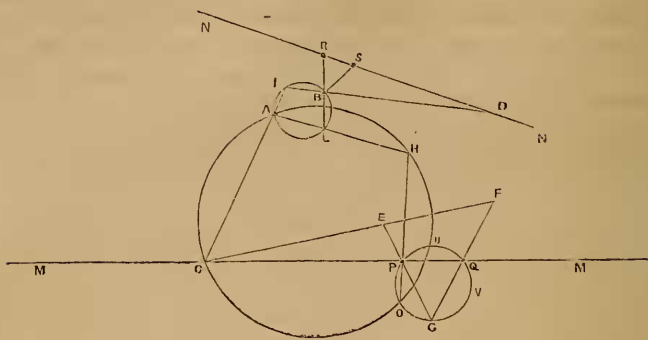
Now, if we suppose R and U to remain fixed, and S to become greater and greater in distance from the fixed points, until it vanishes at infinity, then for points C at a finite distance, we shall have SR = SC, and hence

$$PC.RU = QC.RC.$$

This case of the problem can be expressed as follows:—"*Given three points,* P, Q, R, *in a straight line, to find another* C *in the same, such that* K *being a line of given length (and known sign in respect to the directions on* MN*), we shall have* PC.K = QC.RC."

The solution may evidently be worded thus:—In MN make RU = K; draw RA (not in MN) equal RU; draw PO and QO, making the angles PO right to Q, and QO right to P equal respectively to AR right to U, and UR right to A; produce PO to cut RA in H; describe the circle OHA, and it will cut MN in the required points C, C.

GENERATING PROBLEM TO THE THREE SECTIONS.



Given the points P Q *on a straight line* MM, *and the points* R S *on a straight line* NN; *through two given points* A B *to draw two straight lines* AI, BI, *making the angle* IA *right to* B *of a given angular magnitude* θ *right, and such that* C *and* D *being the points in which* AI *and* BI *cut* MM *and* NN, *we shall have* PC.SD : QC.RD : : l : k; ($\frac{l}{k}$ *being of given magnitude and known sign in respect to the directions on* MM *and* NN).

ANALYSIS.

The circle AIB is evidently known. Suppose we draw a straight line CEF, making the angle CE or CF right to P or Q equal the angle DB right to R or S, and that through P and Q we draw PE and QF to cut it, so that the angle PE right to C = RB right to D, and the angle QF right to C = SB right to D. Then the triangles CPE, CQF are similar to the triangles DRB, DSB; moreover it is evident that from these triangles we have the relation PE.SB : QF.RB : : PC.SD : QC.RD : : l : k.

Let G be the point of intersection of PE and QF.

From the last proportion we have PE to QF in the known ratio of l.RB to k.SB; and therefore (see Porism 4th, in *Transactions* of 1859) the circles EGF, CQF, CPE pass through a point O in the circumference of the circle PGQ, which is such that PO : QO : : l.RB : k.SB; and hence, as PG and QG are known, the point O in the circle PQG is known.

Again, the angle OC right to P = EC right to P = BD right to R; and therefore if L be the other point in which RB cuts the circle AIB, and H that in which OP cuts AL, we have the angle OC right to P or H = BD or BI right to R or L = AI or AC right to L or H; hence a circle can pass through O, H, A, and C; but O, H, and A are known points; therefore the point C in which the circle OHA cuts MM is known, and therefore the point I in which CA cuts the circle AIB, and also the point D in which IB cuts NN.

COMPOSITION.

Through A and B describe the circle AIB, such that I being any point in its circumference, the angle IA right to B = θ right; draw PG and QG, making the angle PG right to M = RB right to N, and the angle QG right to M = SB right to N; describe the circle PGQ, and in it (on the same or opposite side of PQ with G, according as $\frac{PG}{QG}$ and $\frac{l.RB}{k.SB}$ have like or unlike signs) find O such that PO : QO :: l.RB : k.SB; draw RB to cut the circle AIB in L; draw OP to cut AL in H; describe the circle OHA; through either point C in which the circle OHA cuts MM draw CA to cut the circle ABI in I; draw BI to cut NN in D: then will AI and BI be as required.

For through the other point E in which the circle OCP cuts GP, draw CE to cut QG in F.

The angle EC right to P is = OC right to P or H = AC or AI right to H or L = BI or BD right to L or R; therefore, since the angle PC right to E = RD right to B, the triangles PCE, RDB are similar, and CP or CQ right to E or F is equal DR or DS right to B.

And since the angle QG or QF right to P or C is equal angle SB right to R or D, therefore the triangles CQF, DSB are similar. Now from these two pair of triangles we evidently have PE.SB : QF.RB :: PC.SD : QC.RD; but (see Porism in *Transactions* for 1859) we have PE : QF :: PO : QO :: l.RB : k.SB, and therefore PE.SB : QF.RB :: l : k; hence PC.SD : QC.RD :: l : k.

DISCUSSION.

Since there are two points of intersection C, there are two solutions to the problem, both real or both imaginary, according as these points are real or imaginary.

If the ratio $\frac{l}{k}$ be unrestricted as to sign, then it is evident there are two answerable points O, and therefore two circles OHA, and four points C. Moreover, the points O being necessarily on opposite sides of PQ, it follows that two of these four points C must be always real.

Limiting Values for θ Right. *

There are evidently limits to θ right only when A and O are on the same side of MM.

Now the point O is found independent of the magnitude of θ; and it is evident that by describing the two circles through A and O which touch MM, and putting H′ and H′ for the points in which OP cuts them again, and I′ and I′ for the points in which AH′ and AH′ cut RB, then will the angles I′A right to B and I′A right to B be the limiting values for the angles θ right. And if X be the point in which AO cuts I′I′, it is evident that according as $\frac{OH'}{OH'}$ and $\frac{XI'}{XI'}$ have like or unlike signs, so will any straight line through A cutting I′I′ and H′H′ in L and H give $\frac{LI'}{LI'}$ and $\frac{HH'}{HH'}$ of like or unlike sign. Hence it follows that when $\frac{OH'}{OH'}$ and $\frac{XI'}{XI'}$ have like signs, the limiting values *include between them* all values (and no others) of the angle θ right, for which AI and BI are imaginary; and when $\frac{OH'}{OH'}$ and $\frac{XI'}{XI'}$ have unlike signs, the limiting values *have outside them* all values (and no others) of angle θ right, for which AI and BI are imaginary. And when θ right is equal either of the limiting values, the lines AI are coincident and real.

Limiting Values for the Ratio $\frac{l}{k}$.

When the points C may be imaginary, the point O must evidently fall on the same side of PQ with the points A and H. And it is evident we arrive at the limiting values of $\frac{l}{k}$, or $\frac{PO.SB}{QO.RB}$, or (which is equivalent) of $\frac{PO}{QO}$, by finding the points o' and o' in the circle PGQ, so that h' and h' being the points in which Po' and Po' cut AL, the circles A$h'o'$ and A$h'o'$ touch MM.

Moreover, it is evident that the two ratios $\frac{SB.Po'}{RB.Qo'}$ and $\frac{SB.Po'}{RB.Qo'}$ are the required limits, and that they have the same sign, and that according as any value of $\frac{l}{k}$ (having like sign) is of a

* Geometers who do not adopt my improved methods of indicating angles, will find it impossible to define the *limits* of the angular magnitude θ.

magnitude comprehended between them, or equal to one of them, or not comprehended between them, so accordingly will the circle AOH corresponding cut MM in two imaginary points, in two real and coincident points, or two real and distinct points C.

Porismatic Relations of Data.

1. It is evident the problem becomes indeterminate when the circle OAH becomes indeterminate, &c. Now, if the circle PQG passes through A, and that the point O coincides with A, then H on AL will also coincide with A; if, AL does not coincide with OP; and, as the chord OH on OP is equal zero, the circle AHO must be infinitely small. But if AL and OP coincide, then, although AH and OH are "infinitely smalls," they lie both on AL, and therefore it is evident that any circle touching AL in A is an answerable circle AHO. Therefore in this last state of the data the points C are innumerable, and the problem is "porismatic," as well as if we conceived O to move to A having OP parallel AL, and thus causing H to be indeterminate when O coincides with A.

2. If R coincides with S, and that we suppose Q to approach P until it comes to coincide with it, then G is at infinity, and the straight line MM lies in the infinite circumference PQG. And for all values of $\frac{l}{k}$ other than $\frac{1}{1}$ the point O coincides with PQ, and one point C is coincident with PQ, and the other point C with the point in which AL cuts MM. But when $\frac{l}{k} = \frac{1}{1}$, then as the point O, and the point C coincident with O, may be anywhere in MM, the problem is said to be "porismatic."

Remarks concerning Particular Cases.

1. Since PC.SD : QC.RD :: l : k, or, which is the same thing, since

PC.(RD − RS) : (PC − PQ).RD :: l : k, therefore,

when $l = k$, we have

PC.RS = PQ.RD

or PC : RD :: PQ RS

Hence we derive a method of solving the problem—"*Given the points* P, R *in the given straight lines* MM, NN; *through two given points* A *and* B *to draw* AI *and* BI *making the angle* IA *right to* B *of a given magnitude* θ *right, and such that*

C *and* D *being the points in which* AI *and* BI *cut* MM *and* NN, *we shall have* PC *to* RD *in a given ratio of* m *to* n."

It is evident that the points G and O are coincident, and ∴ the solution of this enunciated problem can be worded as follows:—

Take Q and S on MM and NN, so that PQ : RS : : m : n; through A and B describe the circle AIB, which is such that I being any point in it, we have the angle IA right to B = θ right; through P and Q draw PG and QG, making the angles PG and QG right to M, equals respectively to the angles RB and SB right to N, through L where RB again cuts circle ABI, draw AL to cut PG in H; describe the circle AHG; through either point C in which it cuts MM draw CA to cut circle AIB again in I; draw IB to cut NN in D; then will PC : RD : : PQ : RS : : m : n.

2. If we suppose not only $l = k$, but also θ = zero, and B coincident with A; then it is evident the problem becomes the "*Section of Ratio*" of Apollonius. Moreover, it is evident the preceding solution to the Apollonian problem flows directly from the present more general problem, for in this particular state of the data we evidently have G coincident with O, and B, L, and I coincident with A, and ARH in straight line, &c.

3. Since PC.SD : QC.RD : : l : k, if we suppose in NN SR and RU always equal l and k, we have PC.SD : QC.RD : : SR : RU.

Now, if we suppose R and U to remain fixed, and that S becomes infinitely distant, then for a point D at a finite distance, we have SR = SD, and therefore PC.RU = QC.RD.

Hence (QC—QP). RU = QC.RD, and (RU—RD) QC = QP.RU, ∴ DU.QC = QP.RU.

Or, UD.QC = QP.UR = a known magnitude.

Hence, we derive a method of solving the problem. "*Given the points* U *and* Q *in given straight lines* NN *and* MM; *through two given points* A *and* B *to draw two straight lines* AI *and* BI *making the angle* IA *right to* B *equal a given angular magnitude* θ *right, and such that* C *and* D *being the respective points in which* AI *and* BI *cut* MM *and* NN, *we shall have* UD.QC = $m.n$." Where $m.n$ is given in sign, and the directions on the given lines particularised.

It is evident that in this case G coincides with P, and that circle GPQ touches PG at this double point, and ∴ that the triangle QOP is similar to URB, &c.

Hence, the solution of this enunciated problem may be made as follows:—In MM and NN take P and R such that QP.UR $= m.n$; describe the circle AIB such that I being any point in it, the angle IA right to B $= \theta$ right; through P draw a line PG making the angle PG right to M = RB right to N, and describe the circle QOP through Q which touches this line at P; find the point O in this circle such that PO : QO : : RB : RU; through the point L in which RB again cuts the circle AIB, draw AL to cut PO in H; describe the circle AOH; through either point C in which this circle cuts MM, draw CA to cut the circle AIB again in I; draw BI to cut MN in D. Then will AI and BI be answerable lines.

Moreover, it is evident that the angles PO and QO right to M are respectively equals to the angles BR and UR right to U and B, and that we can, therefore, determine PO and QO without drawing PG or describing the circle QPOG.

4. If in addition to the conditions of this third case, we suppose the angle θ = zero, and B coincident with A; then, it is evident, the problem becomes the "*Section of Space*" of Apollonius. Here again it is evident RAH is a straight line, and that the solution which I have given to this Apollonian problem has been derived from the present more general problem.

5. We have PC.SD : QC.RD : : l : k, when the angle θ has any finite magnitude; and it is evident that when θ = zero, and that MM coincides with NN, we have PC.SC : QC.RC : : l : k, which is the "*Determinate Section*" of Apollonius. It is further evident that the solution just given to this Apollonian problem has been derived from the present more general one.

REMARKS.

If K be the other point in which SB cuts the circle AIB; then it is obvious AK and QO intersect in a point T in the circle AHO.

If U be the other point of intersection of the circle OGPQ with any circle AOH, and that PV be drawn parallel to AL to cut circle OGPQ in V, then will VUA be in one straight line. Therefore, as the point U is known independent of the ratio, the limiting circles $Ah'o'$U passing through A and U can be hence easily described.

It is also evident QV is parallel to AK, &c.

Again it may be remarked that if we could solve the generating problem by a different method, we could thence derive other analogous solutions to the 'the Three Problems of Section.'

However, instead of giving another solution to the problem, in which it is required to have PC.SD : QC.RD : : l : k. I will now solve the more extended generating problem in which it is required to have

PC.SD + $p.s$: QC.RD + $q.r$: : l : k, where the magnitudes and signs of the rectangles $p.s$, and $q.r$ are given.

ANALYSIS.

(The figure to be supplied by the reader.)

Suppose we draw PG, QG, making the angle PG and QG right to M, respectively equals to SN and RN right to B. Then the point G is given.

If we draw CE and CF meeting PG and QG in E and F, so that the angles EC and FC right to P and Q shall be each equal to angle DB right to R or S. Then it is evident the triangles QCF, PCE, are similar to the triangles RBD, SBD, and that a circle can pass through CFEG. Moreover, it is evident PC.SD and QC.RD are respectively equal to PE.SB and QF.RB, and therefore we have PE.SB + $p.s$: QF.RB + $q.r$: : l : k.

And if in PE and QF we take the points J and T, such that JP.SB and TQ.RB are equals respectively to $p.s$ and $q.r$; then it is evident the points J and T are known, and that JE.SB : TF.RB : : l : k. Now, from the porisms in *Transactions* for 1859, we know that the circle EFG will cut the known circle JTG in a point O such that JO.SB : TO.RB : : l : k; and therefore O is a known point in circle JTG.

Again, let V be the point in which CA cuts circle CEFG, and U that in which GV cuts circle JTG, and H that in which UA again cuts circle JTGU.

We have angle HO right to U or A = GO right to U or V = CO right to V or A, and therefore a circle can pass through COH and A; but the angle VG right to C being equal EG right to C it is equal DN right to B, and ∴ if W be the point in which GV cuts NN, a circle can pass through VIDW, and the angle WV right to D or N is equal the angle IV or IA right to D or B, and thererefore GW is known in position. Moreover, the point U where GW cuts circle

JTG is known, and the other point H in which UA cuts circle JTGU, and ∴ the point C in which the circle HAO cuts MM, and hence CAI and BID.

The *composition*, &c., may be easily made. However, in order to familiarise the methods of arriving at the limits of angular magnitudes in other questions, I will indicate the nature of the limits of θ right.

Limiting Values for θ Right.

As the points O and A are known independently of θ, ∴ it is evident that by describing the two circles through O and A which touch MM, and putting H′ and H′ for the points in which they again cut circle JTG, and U′ and U′ for those in which AH′ and AH′ again cut this same circle, and W′ and W′ for the points in which GU′ and GU′ cut NN, then will the angles W′G right to N and W′G right to N be the limiting values of θ right. And, if h' and w' be the points in which GA cuts the circle JTG and line NN, it is evident that G may be regarded as a position of U corresponding to w'.

Moreover, it is evident that if w' lies between W′ and W′, and that h' *is not* inside or outside *both* circles through A and O touching MM, then will the circle O Ah' cut MM in imaginary points; and the limiting values for θ right are evidently such *as to include between them* all values of θ right (and no others) for which the lines AI and BI are imaginary. But if w' lies between W′ and W′, and that h' is inside or outside both circles through A and O touching MM, then will the limiting values of θ right be such *as to have outside them* all values of θ right (and no others) for which the lines AI and BI are imaginary.

And it is further evident, that if w' lies outside W′W′ and that h *is not* inside or outside both circles through A and O touching MM, then will the limiting values of θ right be such *as to have outside them* all values of θ right (and no others) for which AI and BI are imaginary; but if w' lies outside W′W′, and that h' is inside or outside both circles through A and O touching MM, then will the limiting values of θ right be such *as to include between them* all values of θ right (and no others) for which AI and BI are imaginary.

When θ right is equal either limit, the lines AI are coincident and real.

Moreover, it is evident that when A and O are not on the same side of MM there are no real limits to angle θ right, &c.

NOTES.

1. It is evident that the *porisms* in the *Transactions* for 1859 can, with many others, be derived from the porismatic states of the data of this or the first Generating Problem of the Three Sections.

2. And very probably, the Greek geometers derived from this and other kindred problems, by means of projections, &c., part of their '*porismatic knowledge*' which is now known as the '*anharmonic properties of pencils and divisions.*'

3. In the investigations of the limits of the angles θ right in problems 1 and 2 of my paper in the *Transactions* for 1859, it would be well to omit all the words from 'And it is moreover evident,' &c., and substitute the following :—And when the circles iBC circumscribe the portion of circle ACH which *is not* within or outside both the circles BAi, the limiting values include between them all values of θ right, and no others, for which CO and BO are imaginary; but when the circles iBC circumscribe the portion of circle ACH which *is* within or outside both circles BiA, then the limiting values have outside them all values of θ right, and no others, for which CO and BO are imaginary. At the limits the lines CO are real and coincident.

THE TANGENCIES.

FIRST SOLUTION.

(See Plate.)

To describe a circle to touch three given circles A, B, C.

ANALYSIS.

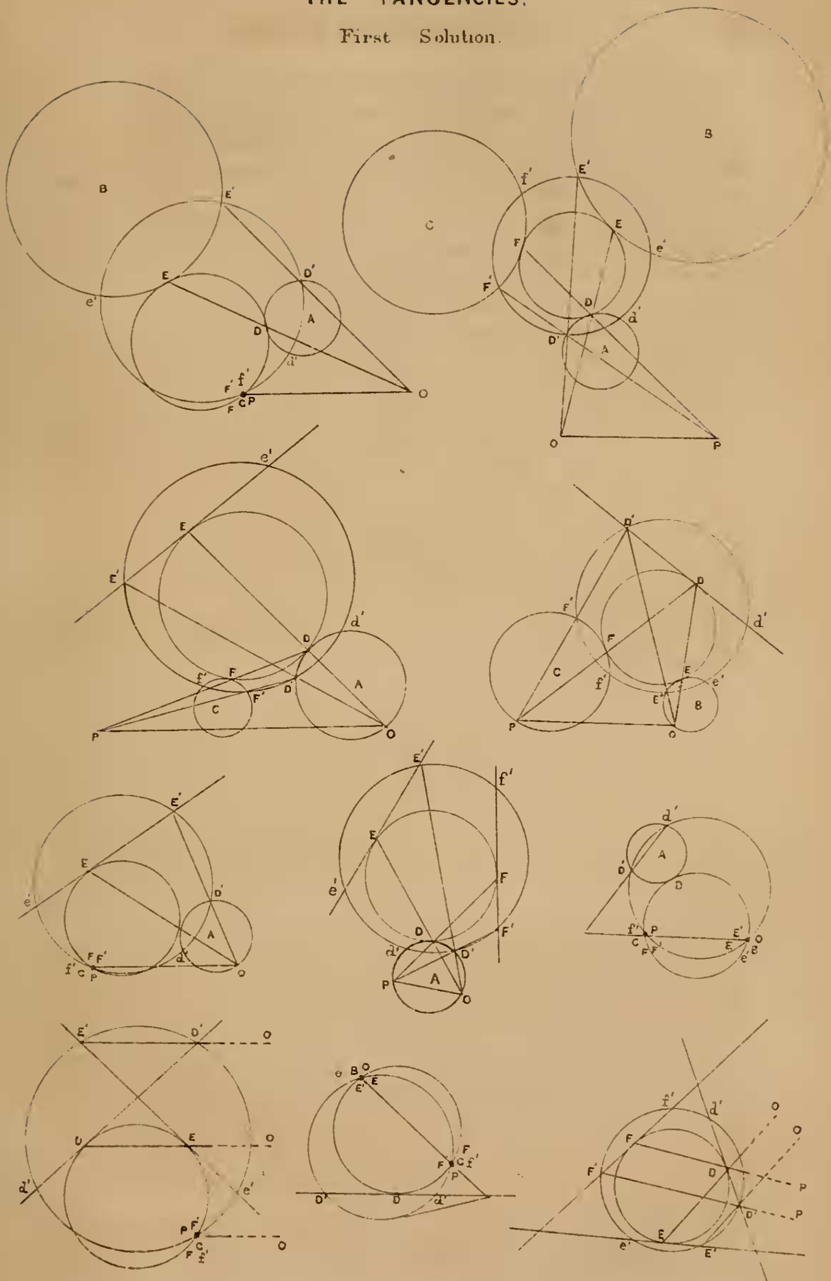
Let D, E, F, be the respective points of contact of the required circle with the circles A, B, C. Then DE passes through O a known centre of similitude of circles A, B; and DF passes through P, a known centre of similitude of the circles A, C.

Now if D′ be any assumed point in circumference A, and that E′, F′, are the dissimilar points in which OD′ and PD′ cut circumferences B and C; then PD′.PF′ = PD.PF, and OD′.OE′ = OD.OE; and it follows that the circles D′E′F′, DEF, have PO as radical axis.

Let d', e', and f', be the other points in which the known circle D′E′F′ cuts the circles A, B, C. It is evident the

THE TANGENCIES.

First Solution.



straight line $D'd'$ and the common tangent to the circles A and DEF at D, cut each other in PO the radical axis of the circles DEF and $D'E'F'$. And for like reasons it is also evident that the intersection of $E'e'$ and the tangent to circle B at E, and also the intersection of $F'f'$ and tangent to circle C at F, are in PO.

But the intersections of the straight lines $D'd'$, $E'e'$, and $F'f'$ with PO are known; ∴ the tangents from these points to the respective circles A, B, C, are known; hence, the points of contact D, E, F, being known, the circle DEF is known.

—— Or, having found either point of contact the others can be easily determined. Thus for instance when D is found, then E and F are the dissimilar points in which OD and PD cut the circles B and C.

COMPOSITION.

Find O a centre of similitude of circles A and B; find P a centre of similitude of circles A and C; through D' any assumed point in circumference of circle A, draw OD' and PD' to cut the circumferences of B and C in the points E' and F' dissimilar to D' on circumference A; describe the circle $D'E'F'$ and draw $D'd'$, $E'e'$, $F'f'$, its respective chords of intersection with the circles A, B, C, to cut the straight line PO in a, b, c.; draw aD tangent to the circle A; draw OD and PD to cut the circles B and C in the points E and F dissimilar to point D on circle A; describe the circle DEF. Then will DEF be a required circle.

For $OD.OE$ being $= OD'.OE'$, and $PD.PF = PD'.PF'$, it follows that OP is the radical axis of the circles DEF and $D'E'F'$, and therefore that aD is tangent to the circle DEF as well as to circle A at the point D: and hence the circle DEF touches circle A at D.

And since ODE passes through the point of contact D of the circles DEF and A, and that O is a centre of similitude of circles A and B, and that the points D and E on circles A and B are dissimilar, ∴ the circle DEF touches the circle B in E. And for similar reasons the circle DEF touches circle C in F.

NOTES.

It is well to observe that we can find the points D and F from E (E being the point of contact of a tangent from b to circle B) by drawing OE to cut the circle A in the point D dissimilar to the point E on circle B, and then PD to cut the circle C in F the point dissimilar to D on circle A. And in like manner we may find D and E from F the point of contact of the tangent from c to circle C.

Moreover, it is evident that we can find D even when the circle A is infinitely great, for the tangents from a to circles A and D′E′F′ are equal.

As there are two points O, and two points P, there are four lines PO and ∴ evidently four answerable points a; and hence, as there are two tangents from each point a to the circle A, it follows that there are in all eight answerable circles DEF, real or imaginary in pairs.

Now DD, EE, and FF, are the polar chords of the circles A, B, and C in respect to the points a, b, c in PO. Let Q be the centre of similitude of the circles B and C through which EF passes. Then OPQ is straight. And as DD must pass through a' the pole of OPQ in respect to circle A, and that EE must pass through b' the pole of OPQ in respect to circle B, and that FF must pass through c' the pole of OPQ in respect to circle C; hence it follows, because ODE, PDF and QFE are straight lines, that DD, EE and FF must meet in the radical centre R of the given circles A, B, C. And this indicates the method of solution given by Gergonne.

It is also easy to see that DD passes through A′ the extremity of the diameter AA′ of the known circle D′Ad'; and EE evidently passes through B′ the extremity of the diameter BB′ of circle BE′e'; and FF passes through C′ the extremity of the diameter CC′ of the circle CF′f'.—Hence other methods of solution.

We may also remark, that as there are four points a, and that the four *polars* of these points in respect to the circle A pass through the radical centre R of the three given circles, it follows that the four points a are situated in the polar of R in respect to the circle A.

And, similarly, the four points b are on the polar of R in respect to the circle B, and the four points c are on the polar of R in respect to circle C.

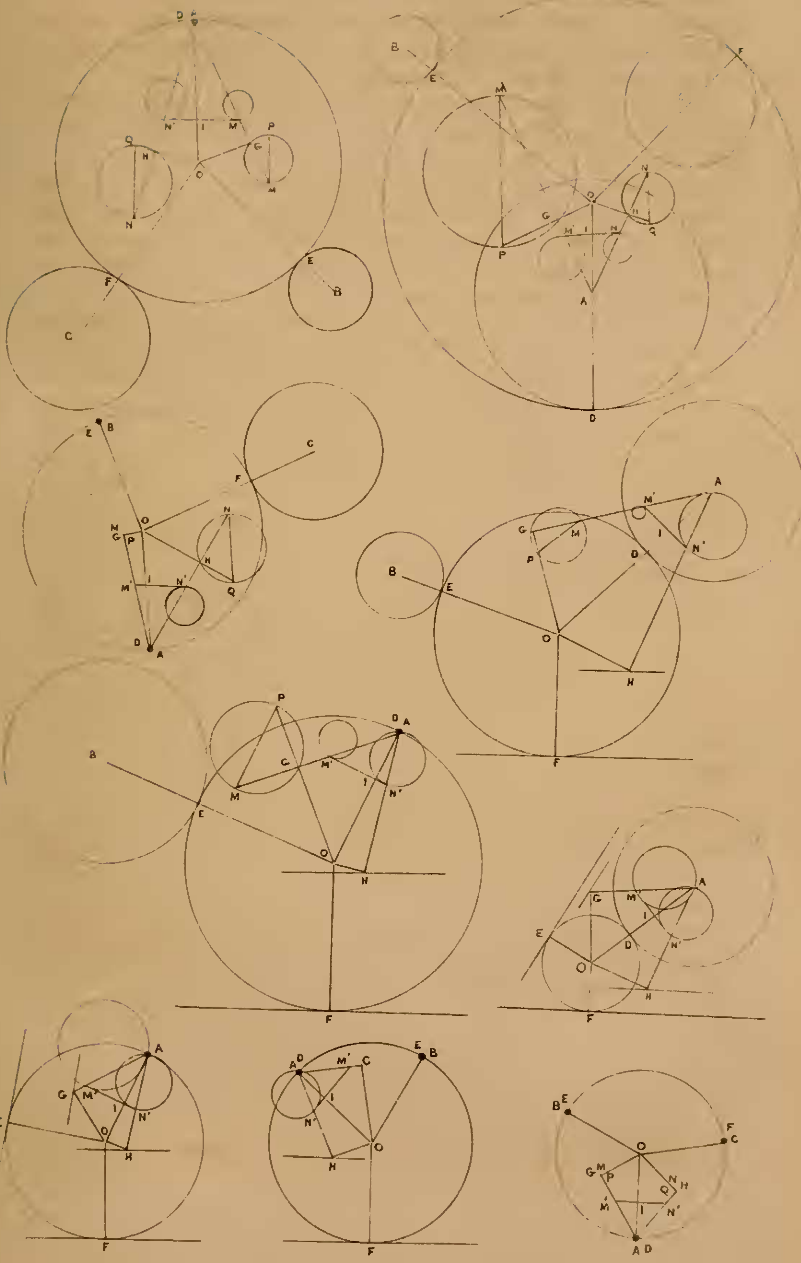
SECOND SOLUTION.

(See Plate.)

To describe a circle to touch three given circles A. B, C.

THE TANGENCIES

Second Solution



ANALYSIS.

Let O be the centre of the required circle, and let D, E, F be its points of contact with the given circles A, B, C. Then OAD, OBE, OCF are straight lines.

Since OD = OE, if OG bisects the angle OD *right round* to OE, and that AG is perpendicular from A on OG, then is the locus of G a known circle having its centre in the middle point of AB, and such that its diameter MP intercepted by AG and OG is parallel to AO. Similarly, since OD = OF, it follows that if OH bisects the angle OD *right round* to OF, and that AH is perpendicular from A on OH, then will the locus of H be a known circle having its centre equally distant from A and C, and such that the diameter NQ intercepted between AH and OH shall be parallel to AO.

And if we assume any auxiliary circle having A as centre, and that AH.AN′ is equal to the square of its radius, then the locus of N′ is a known circle N′H′Q′ such that H′ being the other point in which AH cuts it, the segments cut off by the chords NH and N′H′ are similar. And, for like reasons, if on AG we take the point M′ such that AG.AM′ = the square of the radius of the auxiliary circle, then will the locus of M′ be a circle having with MGP the point A as a centre of similitude, and such that if G′ be the other point in which AG cuts it, the segments cut off from it and circle PGM by the chords MG and M′G′ are similar.

Let I be the point in which AO cuts M′N′.

Since the angles G and H are right, and that AG.AM′ = AH.AN′ = square of radius of auxiliary circle, it follows that M′N′ is perpendicular to AO, and that AO.AI is equal the square of radius of auxiliary circle.

Again, the angle M′N′ right to A being equal to the angle OA right to G = PM right to G, it is ∴ = P′M′ right to G′, and ∴ M′N′ touches circle M′G′P′ in M′. Similarly, since angle N′A right to M′ = OH right to A = QH right to N, it is ∴ equal Q′H′ right to N′, and ∴ M′N′ touches the circle N′H′Q′ at N′.

Hence as M′N′ is a common tangent to two known circles it is itself known, and ∴ the centre O which is the pole of M′N′ in respect to the auxiliary circle, is known, and ∴ also the circle DEF.

And since the radius of the circle PGM can be taken equal to the half sum or half difference of the radii of circles A, B, and that the radius of circle NHQ can be taken equal either the half sum or half difference of the circles A, C, and that to each of the resulting pairs of circles there are two answerable tangents, ∴ it is evident there are four pair of answerable centres O, and ∴ eight solutions to the question which are real or unreal in pairs.

The *composition* may be easily made. And it may be as well to remark that when we suppose the circle C infinitely great, then will the circle NHQ also be infinitely great; and its infinite circumference bisects all straight lines drawn from the point A to the infinite circumference of circle C, &c., &c.

It may be right to observe that by introducing an auxiliary circle into the Fourth Solution in a similar manner to that in this solution, we can make it intelligibly applicable to the minor cases which now escape it; but though this might be an advantage as regards the greater generality obtained, it would not indicate such neat solutions to the leading cases.

THIRD SOLUTION.

(See Plate.)

To describe a circle to touch three given circles A, B, C.

ANALYSIS.

Let D, E and F be its points of contact with the given circles A, B and C.

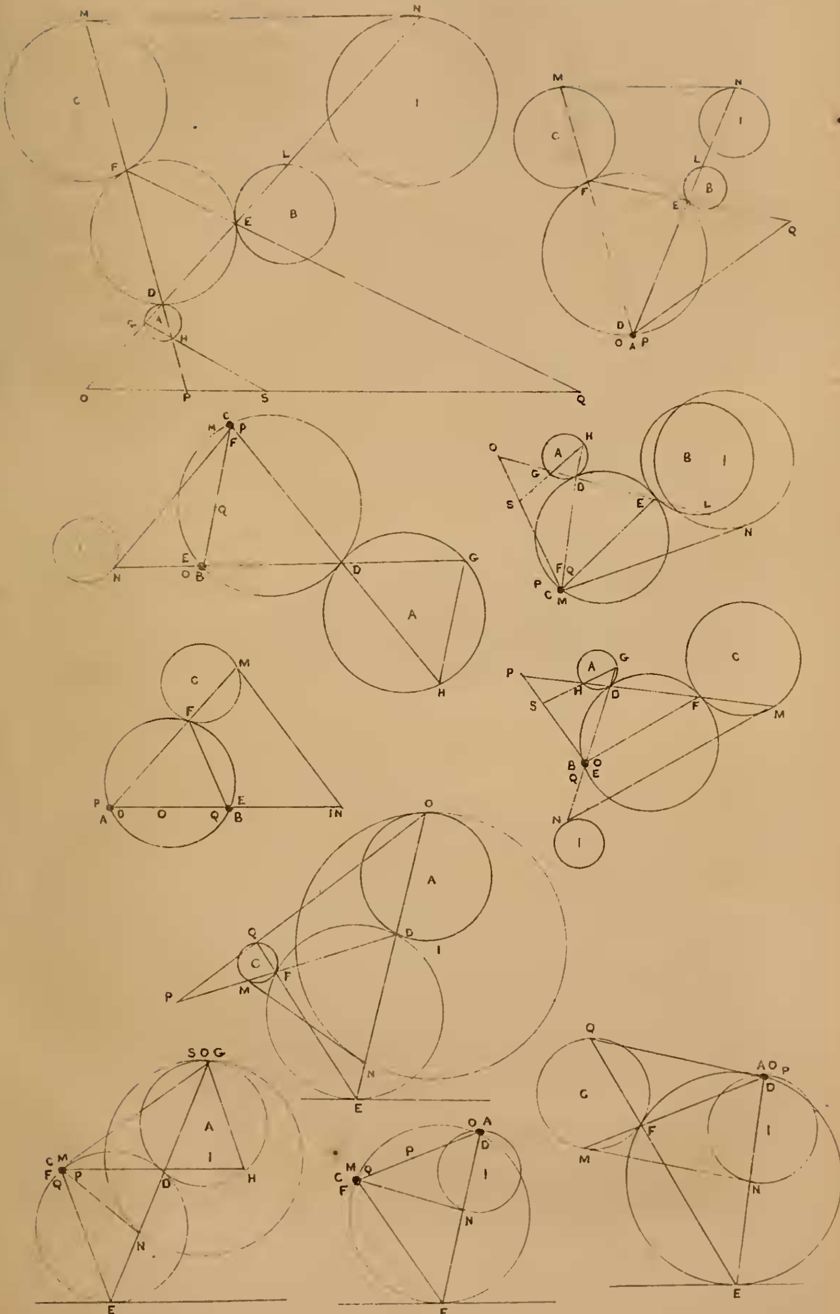
Then DE, DF and EF pass through the respective points O, P and Q, centres of similitude of the given circles which are in one straight line.

Let G and L be the other points in which DE cuts the circles A and B; and let H and M be those in which DF cuts the circles A and C; and let S be the point in which GH cuts the axis of similitude OPQ. Then OS has to OQ the known ratio which OG has to OE, and ∴ the point S is known.

Now the ratio of OG.DH to PH.DG, which is the same as that of OS to PS, is known; and the ratio of PD.PH to OD.OG is also known; ∴ the ratio compounded of these ratios or that of PD.DH to OD.DG is known: and hence as DH : DG : : DF : DE, it follows that the ratio of PD.DF to OD.DE is known.

Or—which amounts to the same—the ratio of OE.FD to PF.ED being the same as that of OQ to PQ is known; and the ratio of PF.PD to OE.OD is (the same with that of

THE TANGENCIES
Third Solution



PF.PM. rad B to OE.OL. rad C) known; hence the ratio of PD.DF to OD.DE is known.

Let N be the point in which a tangent to the circle C at M cuts the straight line ODE.

It is evident DE.DN = DF.DM, and that DE.DN has to OD.DE a ratio compounded of the known ratios of PD.DF to OD.DE, and of DM to PD.

Or—which amounts to the same—it is evident DE.DN has to OD.DE a ratio compounded of the known ratios of OQ to PQ, of PD.PF to OD.OE, and of DM to PD, which may evidently be expressed as the ratio compounded of the ratios of OQ to PQ and of PF.DM to OD.OE.

Hence it follows that the point N must be in the circumference of a known circle NXX having with circle A the point O as centre of similitude.

Moreover, if K be the other point in which DE cuts circle NXX we have the angle XK right to N = angle FE right to D, and ∴ = angle ND right to M; and hence MN is tangent to the circle NXX at N.

Now MN being a common tangent to two known circles, it is itself known; and ∴ the other point F in which the straight line PM cuts circle C is known, as also the point D in which it cuts the circle A similarly to the point M on circle C; and the point E in which ON cuts the circle B similarly to point N on circle NXX is known: and ∴ the required circle DEF is known.

COMPOSITION.

Find O, a centre of similitude of A and B; find P a centre of similitude of A and C; and Q a centre of similitude of B and C in the line OP; and find the point S in OPQ such that OS : OQ : : rad A : rad B.

Through O draw a straight line OD′E′ to cut the circles A and B in dissimilar points D′ and E′; draw PD′ to cut circle C in the point F′ which is dissimilar to D′ on circle A, and to cut it again in M′; find the point N′ in OD′E′ such that D′N′.D′E′ shall have to O′D′.D′E′ the ratio compounded of the ratios of D′M′ to PD′ of OS to PS and of PD.PH to OD.OG, —or which amounts to the same—such that D′N′.D′E′ shall have to OD′.D′E′ a ratio compounded of the ratios of OQ to PQ and of PF′.D′M′ to OD′.OE′; draw N′ I parallel D′A to cut OAB in I; with I as centre and radius IN′ describe a circle; draw MN a common tangent to the circles C and I;

through the point of contact M, on circle C, draw PM to cut the circle C again in F, and to cut the circle A in D similarly to the point M on circle C; draw ON to cut circle B in E similarly to the point N on circle I; describe the circle DEF. Then is DEF a required circle.

NOTES.

This method of solution holds intelligibly good in all cases in which neither of the circles A or C is *infinitely great.* When the circles A and B are *infinitely smalls* the centre of similitude O may have any position whatever in the line AB, as the ratio of their radii may be of any magnitude; and similar remarks apply to the centre of similitude Q when the circles B and C are infinitely smalls. By fixing the ratio of these infinitely small circles, we fix the positions of the centres of similitude; and it is evident we may suppose one of them infinitely small in respect to the other, so as to have the centre of similitude coincident with this other in respect to finite distances. And similar remarks apply as to the ratios of infinitely great radii.

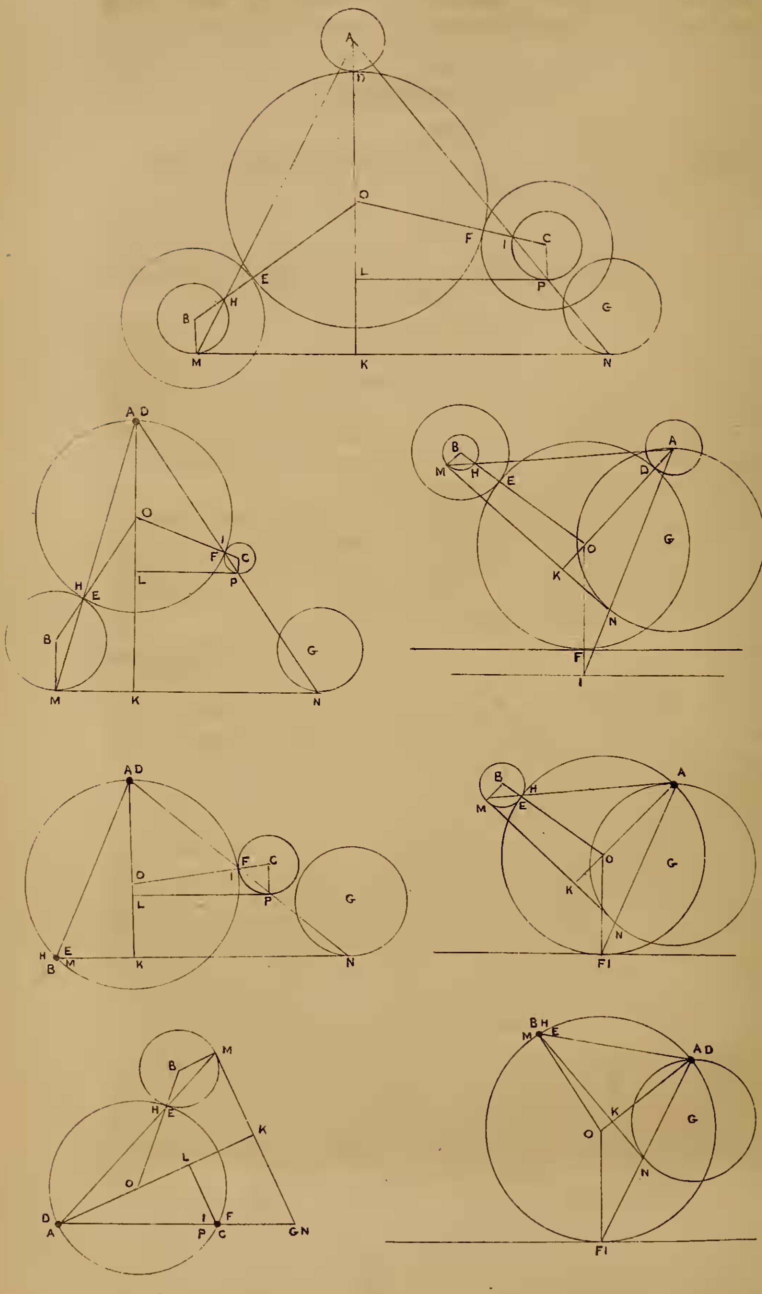
This solution furnishes three methods to the case in which two of the given circles are finite and the third infinitely small. And that one in which we have A the infinitely small circle is in substance the same as what is given by Monsieur Auguste Cauchy.

We are furnished with two methods for the case in which two of the circles are infinitely smalls, and the third finite:—one of which (when A and B are the infinitely smalls), is in substance the same as what is given by *Pappus* as the solution to this case from the *Work* of Apollonius.

However, here as elsewhere, when I speak of a general solution being inapplicable to any case or cases, it is to be considered inapplicable only in a graphical point of view, for a general solution holds mentally good in all cases, even when quantities may be infinitely great or small; and the mind's conviction in such cases is established by its knowledge concerning properties of finite quantities and its own power of legitimately applying the principle of '*continuity*' derived, in degree, from this knowledge.

It is also to be observed that owing to our imperfect knowledge of infinitesimal geometry, or to the nature even of this geometry, it may often happen that we cannot intelligibly arrive at some necessary theorem from one point of view, so

THE TANGENCIES
Fourth Solution



as to employ the steps in a graphic construction; and that, for this reason, it will be compulsory to vary the steps, as is well exemplified in the 3rd solution to the Tangencies.

FOURTH SOLUTION.

(See Plate.)

To describe a circle to touch three given circles A, B, C.

ANALYSIS.

Let O be the centre of the required circle, and let D, E, F be its points of contact with the given circles A, B, C.

In OB take OH = OA, and then EH is = DA; and it is evident the circle having B as centre and BH as radius is known. Moreover, if M be the other point in which AH cuts this circle, and K that in which a tangent to it at M cuts AO, then, since BM and AO are parallels, it follows that MK is perpendicular to AO. It is also evident that AO.AK = $\frac{1}{2}$ AH.AM, and is ∴ of known magnitude.

Similarly, if in OC we take OI = OA, and that from the other point P in which AI cuts the known circle having C as center and CI as radius, we draw a tangent to cut AO in L, then will this tangent be perpendicular to AO, and will AO.AL = $\frac{1}{2}$ the known magnitude AI.AP.

Now if N be the point in which MK cuts AIP, then as AN.AI has to AP.AI the same ratio which AN has to AP or which AK has to AL or which AK.AO has to AL.AO, it follows that AN.AI = twice AK.AO = AM.AH, and ∴ that the locus of N is a known circle G having with circle C the point A as centre of similitude.

And since PL is tangent to C at P, it is evident KN is tangent to circle G at N; ∴, since MKN is common tangent to the two known circles BM and GN, it is itself known; and AK perpendicular to it is known, as also the point O such that AO.AK = $\frac{1}{2}$ the known magnitude AH.AM. Hence the circle DEF is known.

COMPOSITION.

Draw any radius BE′ of the given circle B; draw any radius CF′ of the given circle C; in BE′ and CF′ make E′H′ and F′I′ each equal to the radius of the given circle A; with B and C as centres and BN′ and CI′ as radii describe circles;

draw AH′ and AI′ to cut these circles again in M′ and P′; find N′ in AI′ such that AN′.AI′ = AH′.AM′; draw N′G parallel P′C to cut AC in G; with G as centre and GN′ as radius describe a circle; then, according as $\frac{E'H'}{F'I'}$ has like or unlike sign with $\frac{F'B}{F'C}$, draw MN a common tangent *direct* or *inverse* to the circles BM′ and GN′; draw AK perpendicular to MN and in it find the point O such that AO.AK = $\frac{1}{2}$ AH.AM (this can evidently be done by producing AK until KA′ = AK; and then describing the circle A′M′H′ to cut AK again in O.)

The point O is a centre of a required circle, &c.

NOTES.

Here, too, as in the last solution, it may be remarked that the general solution gives more than one method when applied to many of the particular cases.

From this solution also we arrive at that given by Cauchy for *two circles and a point* (by supposing the circle A infinitely small), and we arrive at that of Pappus given in Leslie's Geometrical Analysis for the case of *two points and a circle* (by supposing the circles A and C infinitely small).

Moreover, we see what has not been remarked by the authors of these solutions to the particular cases, viz.:—that the perpendicular from the point A on the tangent MN passes through the centre O of the required circle.

FIFTH SOLUTION.

To describe a circle to touch three given circles A, B, and C.

ANALYSIS.

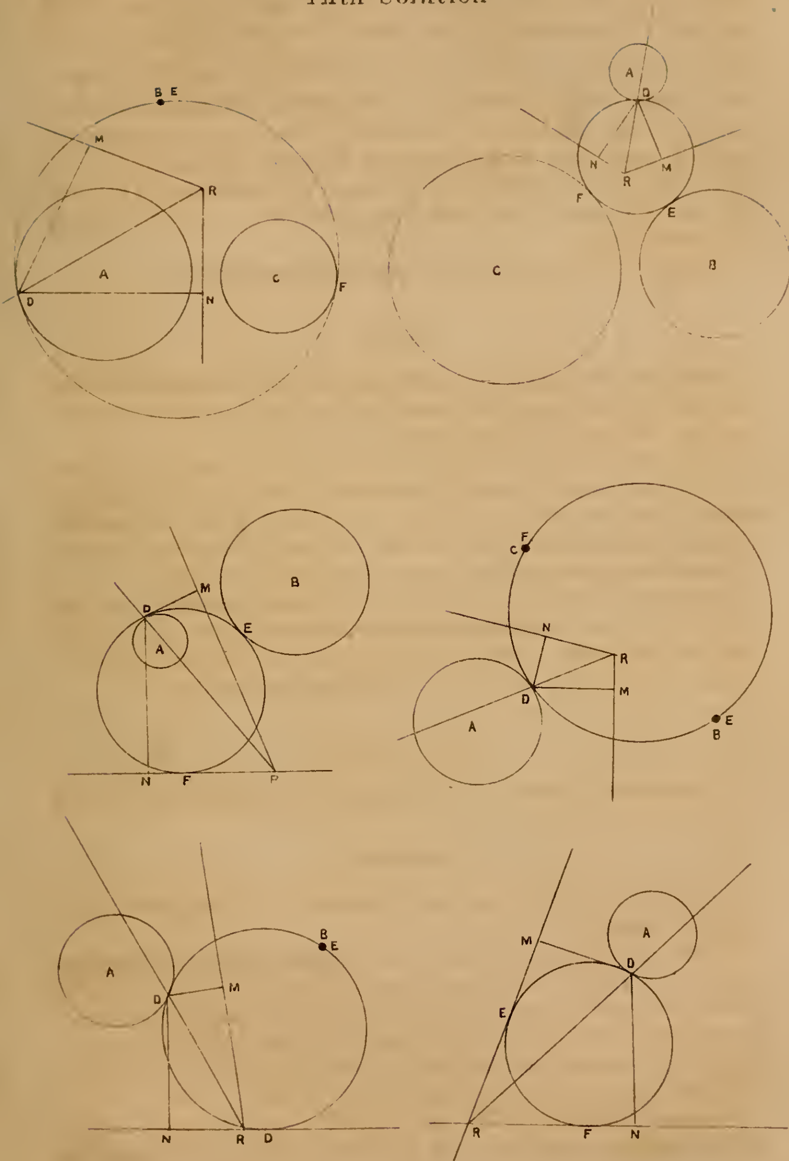
Let D, E, F, be the points of contact of the required circle with A, B, and C.

Now (as will appear from some of the porismatic developments), if DN be a perpendicular from D on the radical axis of the circles A and C, and that DM is a perpendicular on the radical axis of the circles A and B, then will DN.AC have to DM.AB one of the four ratios comprehended in that which $(AC)^2$—(rad A ± rad C)^2 has to AB^2—(rad A ± rad B)^2.

Hence it is evident DN has to DM a known ratio; and ∴ as the radical axes RN and RM are known, it follows that

THE TANGENCIES

Fifth Solution



the straight line RD is known in position; and ∴ the point D where it cuts circle A is known.

Similarly, by drawing perpendiculars from E on the radical axis RM and on the radical axis to the circles B and C, it can be shown that RE through R is known in position, and ∴ E is known.

And in like manner we can find the point F on circle C. Or the circle DEF can be easily found from any of the points D, E, F, of contact, since the lines joining these points pass through known centres of similitude, &c.

The *composition* may be easily made.

NOTES.

The ratio which DN has to DM, is as has been indicated,

$$\frac{DN}{DM} = \frac{AB\left\{AC^2 - (\text{rad } A \mp \text{rad } C)^2\right\}}{AC\left\{AB^2 - (\text{rad } A \mp \text{rad } B)^2\right\}}$$

And, in order to show that this holds good for all values of the radii C and B from zero to infinity inclusive, let c and b be the points in which AC and AB cut the circles C and B, and let a be that in which either of these lines cuts circle A.

We have AC = Ac + cC, rad A = Aa, rad C = Cc, rad B = Bb; and ∴ we can put the ratio under the form

$$\frac{DN}{DM} = \frac{AB\left\{Ac^2 + 2\,Ac.cC - Aa^2 \pm 2\,Aa.cC\right\}}{AC\left\{Ab^2 + 2\,Ab.bB - Aa^2 \pm 2\,Aa.bB\right\}}$$

which evidently holds good for all values of the radii B and C.

The above method of solution requires one circle (as A) to be finite. When circles C and B are infinitely smalls

$$\frac{DN}{DM} = \frac{AB}{AC}\cdot\frac{Ac^2 - Aa^2}{Ab^2 - Aa^2} = \frac{AB}{AC}\cdot\frac{AC^2 - Aa^2}{AB^2 - Aa^2}$$

When circles C and B are infinitely greats

$$\frac{DN}{DM} = \frac{Ac \pm Aa}{Ab \pm Aa}.$$

When c is infinitely great and b = zero

$$\frac{DN}{DM} = \frac{2.AB\,(Ac \pm Aa)}{AB^2 - Aa^2}$$

When c = zero, and b infinitely great

$$\frac{DN}{DM} = \frac{AC^2 - Aa^2}{2.AC\,(Ab \pm Aa)}.$$

From Gergonne's solution, and the theorem on which the above solution depends, it is evident we have expressions for the ratios of the perpendiculars from the poles of the axes of similitude on the radical axes, &c., &c.

SIXTH SOLUTION.

(See Plate.)

To describe a circle to touch three given circles A, B and C.

ANALYSIS.

Let D, E, and F be the points of contact of the required circle with the three given circles A, B, and C.

Then DE, DF and EF pass through O, P and Q centres of similitude of the given circles; and OPQ is an axis of similitude.

Let G and H be the other points in which ED and FD cut the circle A, and S that in which HG (which is parallel EFQ) cuts POQ.

Then OS has to OQ the known ratio which OG has to OE, and which radius A has to radius B, and ∴ S is a known point.

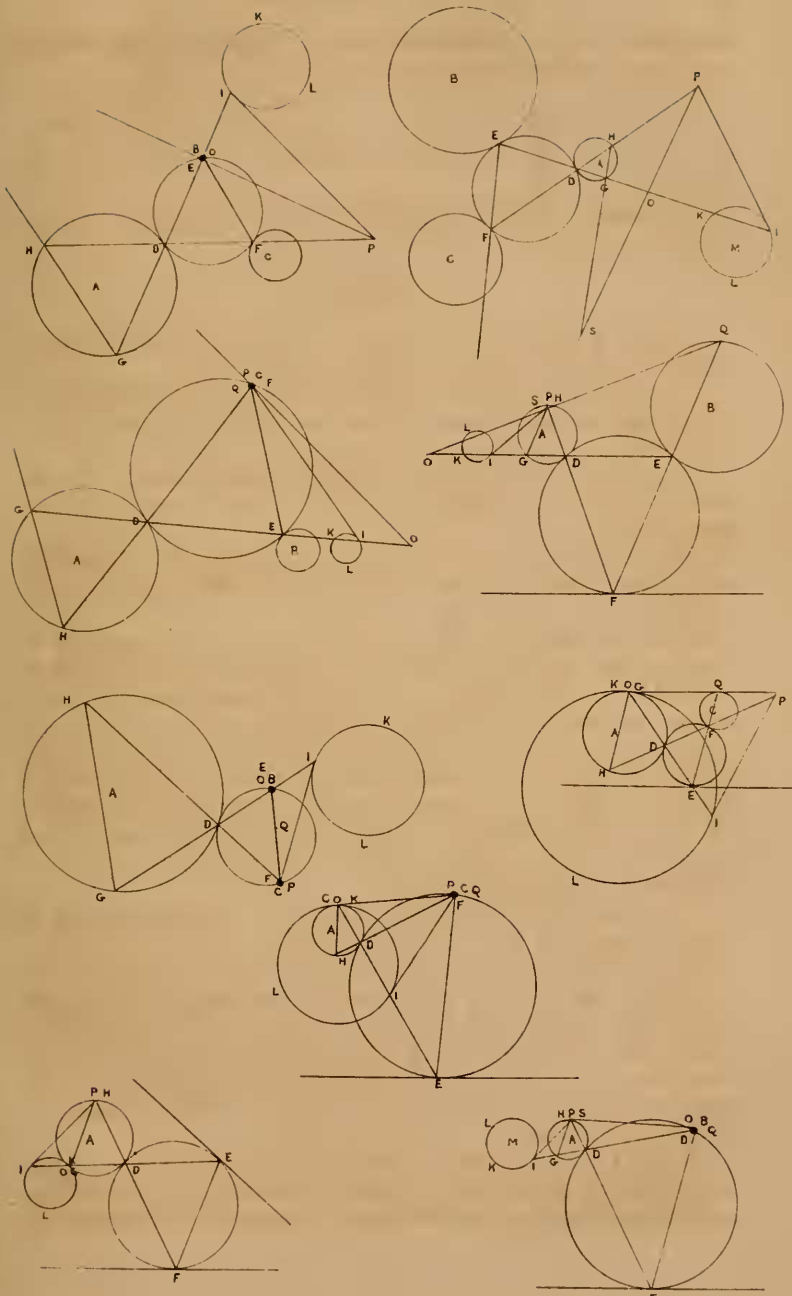
Now the ratio OG.DH to PH.DG, being the same with that of OS to PS, is known; and the ratio of PD.PH to to OD.OG is also known; ∴ the ratio compounded of these ratios, or that of PD.DH to OD.DG is known: or—which amounts to the same—the ratio of OE.FD to PF.ED being the same with that of OQ to PQ is known; and the ratio of PF.PD to OE.OD is known; and ∴ the ratio compounded of these ratios, or that of PD.DF to OD.DE is known; and ∴, as DF : DE : : DH : DG, it follows that the ratio of PD.DH to OD.DG is known.

Let I be the other point in which a circle through P, H and G would cut OGD. Then PD.DH = ID.DG; and ∴ ID has to OD the known ratio which PD.DH has to OD.DG, and the point I must be in the circumference of a known circle ILL having with circle A the point O as a centre of similitude. Moreover, if K be the other point in which DO cuts this circle, then as the angle LK right to I is = HG right to D, it is = ID right to P; and ∴ PI is a tangent to the circle IKL at I.

Now PI is known, and ∴ also IO and the point D on circle A similar to I on circle ILK, as also the point E on B dissimilar to D on A; and the point F on circle C in which PD

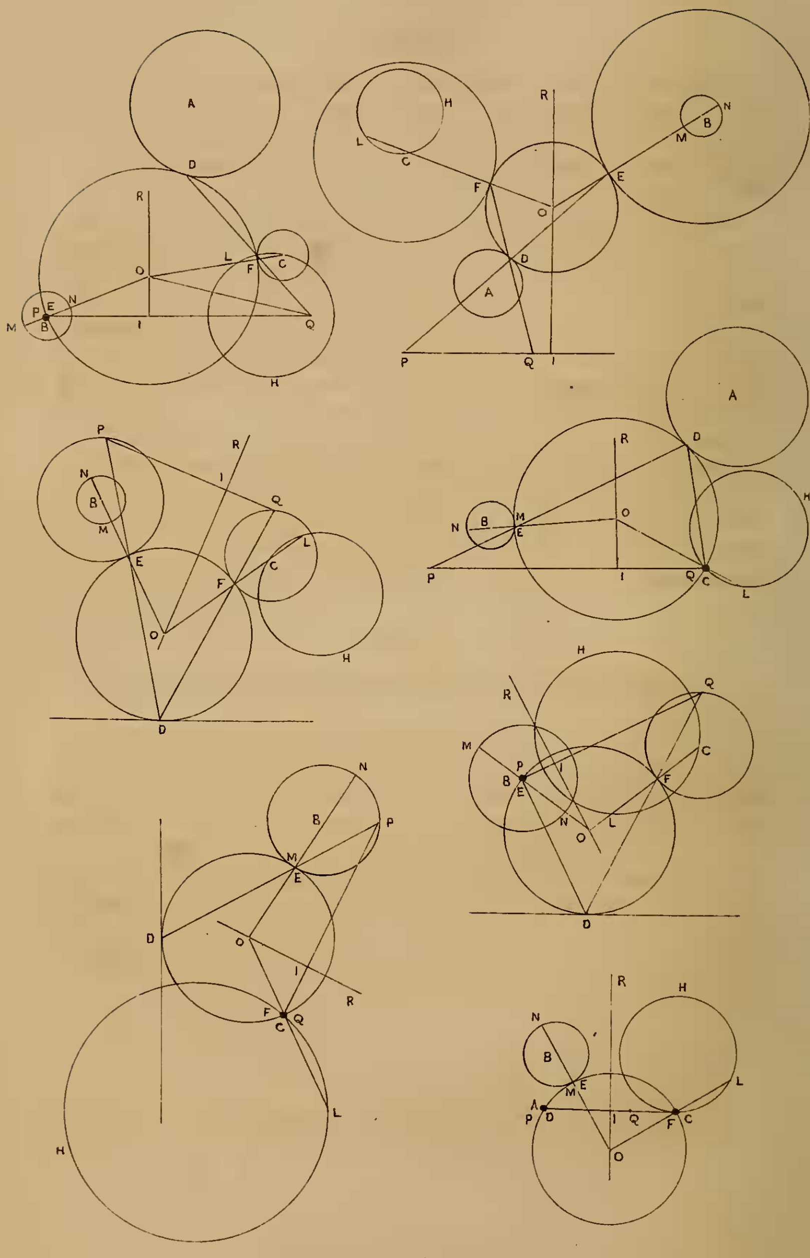
THE TANGENCIES

Sixth Solution



THE TANGENCIES

Seventh Solution



cuts it dissimilarly to D on circle A is known. Hence the circle DEF is known.

COMPOSITION.

Find O a centre of similitude of the circles A and B; find P a centre of similitude of the circles A and C; find Q the centre of similitude of the circles B and C which is in line with O and P; and find the point S in POQ such that PQ : PS : : rad C : rad A.

Take any point D′ on the circumference of circle A, and draw OD′ and PD′ to cut the circles B and C in the points E′ and F′ dissimilar to D′ on circle A; then on OD′E′ find the point I′ such that I′D′ shall have to OD′ the ratio compounded of the ratios of OS to PS, and of PD.PH to OD.OG: or—which is the same—find I′ such that I′D′ shall have to OD′ the ratio compounded of the ratios of OQ to PQ and of PF.PD to OE.OD.

Draw I′M parallel to AD′ to cut AO in M; from M as centre and with MI′ as radius describe a circle, to which draw PI a tangent; draw IO to cut circle A in D similarly to I on circle M, and to cut circle B in E dissimilarly to D on A; draw PD to cut circle C in F dissimilarly to circle A in D; describe the circle DEF. Then is DEF a required circle.

NOTES.

This solution holds for all the cases in which the circle A is finite, &c.

If we were to draw DR tangent to the circle A at D to cut PO in R; then as DR is parallel to PI, and that RP has to RO the known ratio which ID has to OD, it follows that the point R on PO is known; and ∴ the tangent RD to circle A is known; and hence, &c.—another method of solution.

Or we might solve the problem in a similar manner to that of the third solution from the knowledge that DF.DM has to DE.DL the known ratio of AC^2 —(rad A$\mp$ rad C)^2 to AB^2—(rad A $\mp$ rad B)^2 where M and L are the other points in which PD and OD cut the circles C and B.

SEVENTH SOLUTION.

(See Plate.)

To describe a circle to touch three given circles A, B, C.

ANALYSIS.

Let O be the centre of the required circle, and D, E, F the points in which it touches the given circles A, B, C.

Then ED passes through P a centre of similitude of the circles B and A; and FD passes through Q a centre of similitude of the circles C and A.

And since the rectangles PD.PE, QD.QF are of known magnitudes, and that they are respectively equal to $PO^2 - OD^2$ and $QO^2 - OD^2$, $\therefore$ $PO^2 - QO^2$ is of known magnitude and sign, and the locus of O is a known straight line OI perpendicular to PQ, and such that $PI^2 - QI^2 = PO^2 - QO^2$.

Now if in OB we have OM = OC, then EM = the radius of circle B; and BM is of known magnitude; and the circle having B as centre and BM as radius is known.

Let N be the other point in which OM cuts this last mentioned circle; and suppose the circle CHL passing through the point C and having with the circle MN the line IO as radical axis.

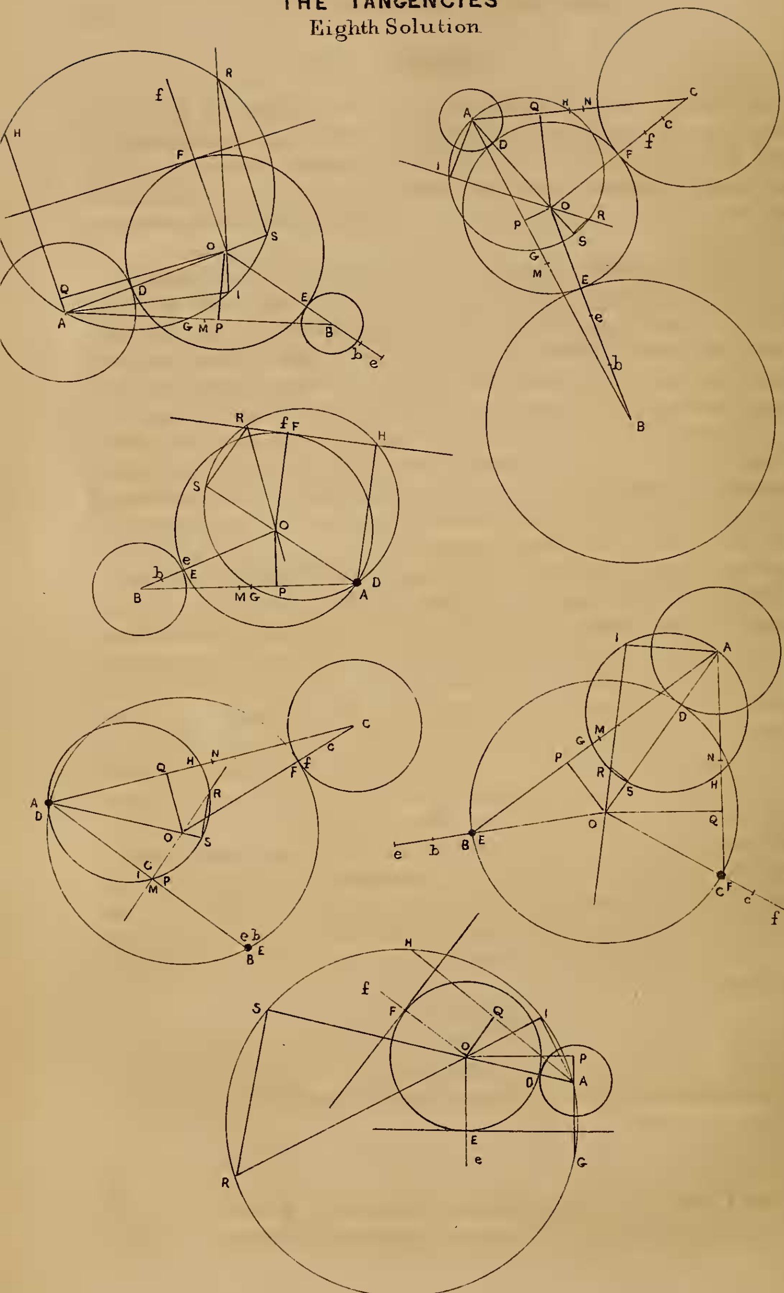
Then L being the other point in which OC cuts the circle CLH, we have OC.OL = OM.ON; hence as OC = OM, it follows that CL is equal the diameter MN, and $\therefore$ of known magnitude. But the circle CHL is known; $\therefore$ CL is known in position, and hence the point O in which it cuts IO, and therefore the circle DEF is known.

COMPOSITION.

Find P a centre of similitude of the given circles A and B; find Q a centre of similitude of the given circles A and C; draw a straight line PD′E′ cutting circles A, B, in dissimilar points D′ and E′; draw a straight line $Qd'f'$ cutting the given circles A, C, in dissimilar points d' and f'; find the point I in the line PQ such that $PI^2 - QI^2 = PD'.PE' - Qd'.Qf'$, and through I draw a straight line IR perpendicular to PQ: draw any radius Be' of the circle B, and from e' in the proper direction on $e'B$ make $e'm'$ = the radius of circle C; with B as centre and Bm' as radius describe a circle, and produce $e'm'$ to cut it in n'; through the point C describe the circle CHL which with circle $Bm'n'$ has IR as radical axis, and in it inflect the chord CL = to the diameter $m'n'$, and let O be the point of intersection of CL with IR: then will O be a centre of a required circle.

THE TANGENCIES

Eighth Solution



NOTES.

Since there are two points P and two points Q, there are four lines IR, and as there are four corresponding circles CL and two chords CL in each, it is obvious there are eight answerable circles O real or unreal, in pairs, according as the circles LCM are greater and less than the corresponding circles $m'n'$.

It is evident that when the circles A and C are infinitely small, and B finite, then may the point Q have any position whatever in the straight line through A and B (because the infinitely small circles may have any ratio whatever just according as we suppose two circles to have any finite constant ratio during their diminution to the infinitely small state.)

This solution does not readily apply to the case in which two of the given circles are supposed infinitely great, or replaced by straight lines; but the following is an *analysis* of a solution which will embrace all the cases in which we suppose the circle A of finite magnitude.

Since the rectangles PD.PE and QD.QF are known in signs and magnitudes, it follows (from one of a class of porisms to be included in subsequent *developments*) that we know the two points o', o', of real or imaginary intersection of all circles having their centres in PQ and respective radii equal the tangents from them to circle DEF. And since the circle A and straight line PQ are known, we know the two points a', a', of real or imaginary intersection of all circles having their centres in PQ and respective radii equal to the tangents from them to circle A. And it is evident we know the circle through the four points $o'o'a'a'$, and that its centre R is in PQ; moreover it is evident the circumference of this circle R passes through D; and ∴ D on circle A is known, and hence AD and the point O in which it cuts the straight line through $o'o'$; and ∴ the circle DEF is known

EIGHTH SOLUTION.

(See Plate.)

To describe a circle to touch three given circles A, B, C.

ANALYSIS.

Let O be the required centre, and let D, E, and F be the points of contact with the given circles A, B, and C.

Since the rectangle under the half sum and half difference of the sides of a triangle is equal to the rectangle under the half sum and half difference of the segments of the base made by a perpendicular from the vertex; $\therefore$ it is evident, that if OP be perpendicular to AB, and M be middle point of AB, and that in OB we have Oe = OA, and b the middle of eB, then will Ob.Bb = AM.MP; and hence Ob has to PM the same known ratio which AM has to bB.

Now if in AB we find the point G such that eb is to GM in the known ratio of Ob to PM, then will Oe or its equal AO have to PG the same known ratio.

For like reasons it is evident that if OQ be perpendicular to AC, and N the middle point of AC, and that we assume Of in OC and = to OA, and that c is middle of fC, and that we find H in AC such that fc shall have to HN the known ratio of AN to Cc, then will AO have to QH this same known ratio.

Now the points G and H are known, and the ratio of PG to QH is known (because AO has known ratios to PG and QH): hence the point I in which the circle AHG cuts the circle QAP is known; and since the angles P and Q are right, it follows that the straight line IO perpendicular to AI is known, as also the point R in which it again cuts the circle AGH.

Again, GR being perpendicular to AG, it is parallel to PO; and PG has to OR a known ratio; therefore AO has to OR a known ratio.

If S be the point in which AO again cuts the circle AGH, it follows, from similar triangles, that AI has to RS the known ratio which AO has to OR, and $\therefore$ the chord RS is of known magnitude, and $\therefore$ also it is known in position; and AS is known, and also the point O where it cuts IR, and $\therefore$ the required circle is known.

COMPOSITION.

Through the centres B and C draw BE$'$ and CF$'$ any two radii of the circles B and C; from E$'$ in either direction on EB make E$'e'$ = radius of circle A; from F$'$ in either direction on F$'$C make F$'f'$ = radius of circle A; bisect e'B in b', Cf' in c', AB in M, and AC in N; find G in AB such that $e'b'$: GM : : AM : $e'b'$; and in AC find the point H such that $f'c'$: HN : : AN : $f'c'$; assume any straight line x and find lines y and z such that x : y : : AM : $e'b'$, and x : z

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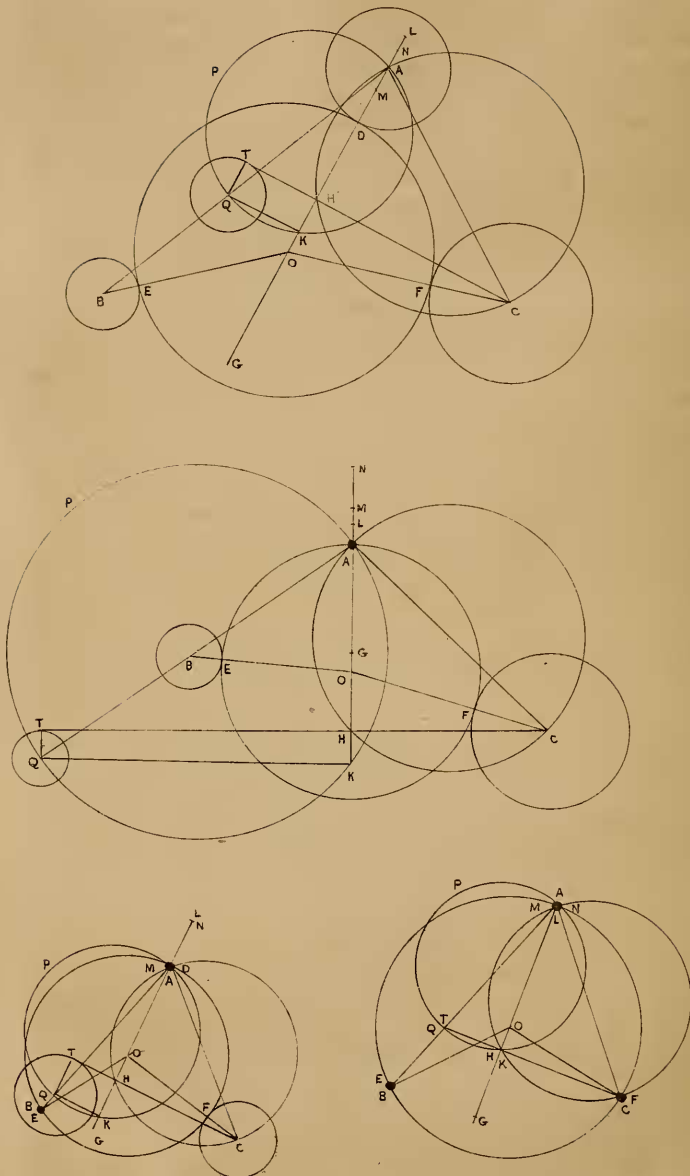
Ninth Solution



: : AN : : $f'c'$; on AB and AC make Gp and Hq equals to y and z, and describe the circles AGH, Apq; through the point of intersection I of these two circles draw IR perpendicular to AI to cut the circle AGH again in R; find the point i in IR so that AI : Ii : : AM : $e'b'$, and draw ii' parallel to AG to cut AI in i'; with R as centre and radius equal ii' describe a circle, and from either point S in which it cuts the circle AGH draw AS to cut IR in O.

Then will O be the centre of a required circle.

NOTES.

This method of solution holds good in all cases in which circle A is not supposed infinite, or—which amounts to the same thing—it holds good in all cases but those in which we suppose the three given circles replaced by straight lines.

If we suppose circle C infinite, then it is evident AC is parallel to OC and perpendicular to tangible portion of the infinite circumference; moreover since the ratio of AN to fc is then one of equality, so will that of AO to QH be one of equality, and ∴ QH will be = Of, and the point H in the perpendicular from A on the known portion of the infinite circumference is known as it is at a distance = radius of circle A therefrom.

It may also be observed that the line IOR is identical with the line IOR of the seventh solution.

NINTH SOLUTION.

(See Plate.)

To describe a circle to touch three given circles A, B, C.

ANALYSIS.

Let O be the centre of the required circle, and let D, E, F be its points of contact with the given circles A, B, C.

If in OA we suppose OM taken equal OB and so that OB and OM have like directions in respect to the directions OE and OD, then DM = EB, and it is evident AM is of known magnitude.

And since OB is equal OM and that $2.OA.OM = OA^2 + OM^2 - AM^2$.

$\therefore 2.OA.OM = OA^2 + OB^2 - AM^2$ both in sign and magnitude. But if G be the point in which the circle having

BA as diameter cuts AO, we have $2.OA.GO = - OA^2 - OB^2 + AB^2$. And from these two we get, by adding equal quantities, the relation $2OA.GM = AB^2 - AM^2$.

Similarly if in OA we suppose ON equal to OC and the like direction in respect to direction OD which the direction OC has to OF, then will $DN = FC$, and will AN be of known magnitude. And it is evident that if H be the point in which the circle having CA as diameter cuts AO, we have in like manner the relation $2OA.HN = CA^2 - AN^2$.

But the ratio of the known quantities $AB^2 - AM^2$ and $CA^2 - AN^2$ is known : $\therefore$ it follows that the ratio of GM to HN which is the same with it is known.

Now if in MN we suppose NL so taken that MA has to NL the known ratio which GM has to HN; then GA has to HL the same ratio.

Hence if in AO we suppose $KA = HL$ and in like direction to it, then GA has to KA a known ratio, and $\therefore$ the point K must be in the circumference of a known circle AKP passing through A and having its diameter AQ in AB.

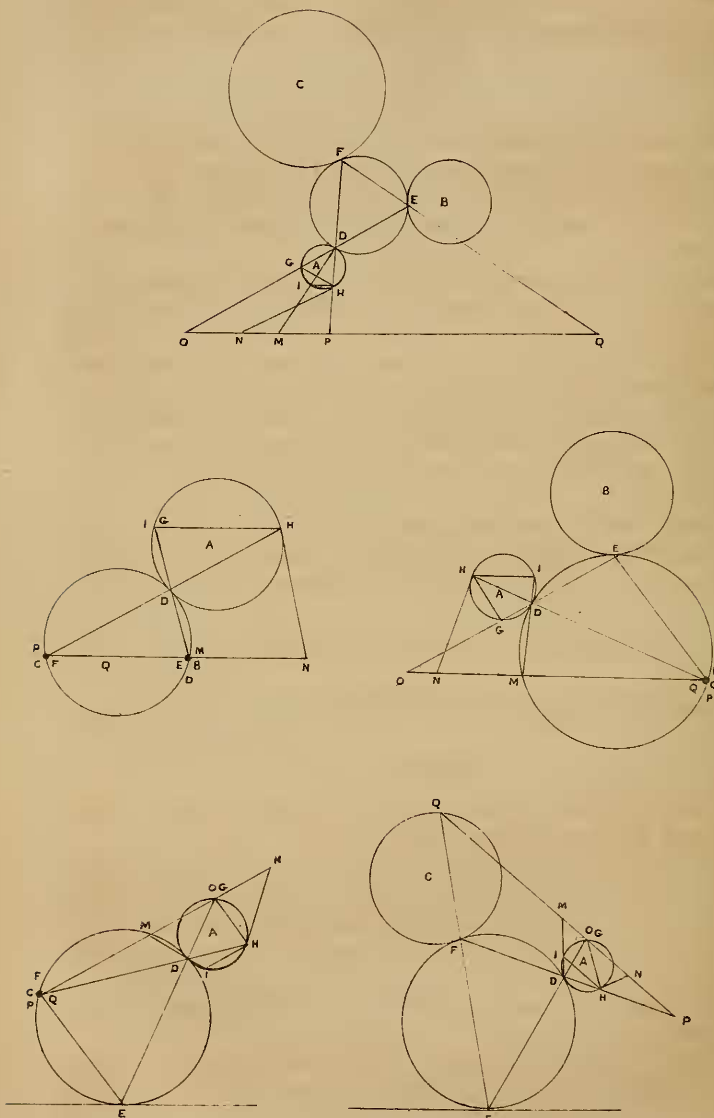
Again since $KA = HL$, we have $KH =$ the known magnitude AL. And the angles CHK, QKH, being right, it follows that QT the perpendicular from Q on CH is equal to KH, and that CT is tangent to the circle having Q as centre and QT as radius; but this circle is known; therefore the tangent CT is known, as also the other point H in which it cuts the circle on AC as diameter; and therefore AHO is known in position.

And, since AHO is known in position, the point N is known; and $\therefore$ as $OC = ON$, the point O is known, and hence the circle DEF.

COMPOSITION.

Through the centre A draw any radius AD′; in D′A take $D'M' =$ radius B, and $D'N' =$ radius C; in M′N′A find the point L′ such that M′A shall have to N′L the ratio which $AB^2 - AM'^2$ has to $CA^2 - AN'^2$ (taking signs into account); in AB take AQ so that AB : AQ : : M′A : N′L (taking note of signs); on AC and AQ as diameters describe circles; with Q as centre and a radius equal to AL′ describe a circle, and draw CT a tangent to it from C; through the other point H in which CT cuts the circle on AC as diameter, draw the straight line AH to cut the circle on AQ as diameter in K; make $HL = KA$ (and in the same direction); make in the

THE TANGENCIES
APOLLONIUS' ORIGINAL SOLUTION.



same line AN = AN′ so that $\frac{AL}{AN}$ and $\frac{AL'}{AN'}$ have like signs; find O in AL so that OC = ON; then from the point O as centre and the point D, when NO cuts circle A (so that ND = N′D′) as distance, describe a circle : this circle will be as required.

NOTES.

This method of solution is not intelligibly applicable to those states of the data in which any of the given circles is supposed infinitely great, or replaced by a straight line; but for the other cases it is thoroughly complete and deserves attention.

Indeed, I may remark that there are many very good solutions applying only to the cases in which none of the given circles A, B, C, is infinite, or when two are infinite. Yet it must not be lost mind of, by those who would succeed, that the solutions to general questions are often arrived at from considering them under some particular states of the data, and divining what modifications are necessary so as to make the solutions which may be arrived at applicable to the more general cases.

It may also be remarked that a theorem evolved in the above solution is directly applicable in a solution to the principal case of the '*Inclinations*' of Apollonius.

APOLLONIUS' ORIGINAL SOLUTION.

(See Plate.)

To describe a circle to touch three given circles A, B, and C.

ANALYSIS.

Let D, E, and F be the points of contact of the required circle with the given circles A, B, and C.

Then, DE passes through O a centre of similitude of the circles A and B; DF passes through P a centre of similitude of the circles A and C; and EF passes through Q a centre of similitude of the circles B and C; and the points O, P, and Q are in a straight line.

Let G and H be the other points in which DE and DF cut the circle A.

Through H draw a parallel to OPQ; and through the other point I in which it cuts the circle A, draw DI to cut OPQ in M.

Since HI and HG are parallel to FEQ and QPO, the angle QO right to E = angle HI right to G and is ∴ = angle DI right to G or DM right to O or E; and ∴ a circle can pass through MDEQ.

But O being a centre of similitude to the circles A and B, the magnitude of OD.OE is known; and ∴ since OM.OQ = OD.OE, the point M is known.

Let N be the point in which a tangent to circle A at H cuts POQ.

The angle NH right to P is = HN right to I and ∴ = DH or DP right to I or M. Hence a circle can pass through DH, M, and N. And since PM.PN is = the known magnitude PH.PD, it follows that the point N is known.

Now N being a known point, the tangent NH to circle A is known; and the point F in which PH cuts the circle C similarly to H on circle A is known; as also the other point D in which it cuts circle A. And the point E on circle B, in in which OD cuts it dissimilarly to D on circle A, is known: and hence the circle DEF is known.

Or, we might determine D by drawing a tangent at D to cut OPQ, &c., because circle MDP touches A at D.

COMPOSITION.

Find O a centre of similitude of the circles A and B; find P a centre of similitude of the circles A and C; and find Q the centre of similitude of the circles B and C which is in line with O and P.

Through O draw a straight line to cut the circles A and B in dissimilar points, D′ and E′; describe the circle QE′D′; through P draw a line to cut the circle A in points H′, D′; through H′, D′, and the other point M in which the circle QE′D′ cuts the line OPQ, describe a circle; through the other point N, in which the circle D′H′M cuts the line QPO, draw NH a tangent to the circle A; draw PH to cut the circle A again in D, and to cut the circle C in F dissimilar to D on A; draw OD to cut circle B in E dissimilar to D on circle A; describe the circle DEF.

Then is DEF a required circle.

NOTES.

This method of solution is intelligibly applicable in a direct manner only when the circle A is finite, and C neither an infinitely great circle, nor infinitely great in respect to the circle B. The reason of such restriction arises from the peculiar nature of infinitesimal geometry causing the indicated operations to be graphically impracticable though mentally possible.

However as regards the five principal cases of the problem; viz., when the circles A, B, C are finite—when A and B are finite and C infinitely small—when A is finite and B and C infinitely small—when A and C are finite and B infinitely great—and when A is finite, C infinitely small and B infinitely great, this solution is remarkably elegant, and depends on very simple well known elementary truths.

That it is in substance the same as the one given by Apollonius, may be easily gathered from Pappus' commentaries on the writings of the celebrated Greek geometers.

He observed that the Apollonian solution to the Tangencies was of such a nature as to indicate a method of inscribing a triangle in a given circle, whose sides would pass through three given points in a straight line. And then, evidently, in order to prepare for a construction to the general problem of inscribing, in a given circle, a polygon, whose sides should pass through given points, he gives the indicated method of solution to the particular case just mentioned, both when the three fixed points are at finite distances from each other, and when one of them is at an infinite distance.

Now, in the solution just given in the text, nothing would be more apt to suggest itself than the fact that GH, a parallel to FEQ, cuts OPQ in a point S, such that OS has to OQ the known ratio which OG has to OE, or which rad. A has to rad. B; and that we could ∴ solve the problem :—"Being given three points O, P, S, in a straight line; to inscribe a triangle DGH in a given circle so that its sides will pass through these points."

And the method which the present solution indicates is exactly the same as is given in Pappus' Mathematical Collections, as well when the three points are at finite distances as when one of them is at an infinite distance.

These coincidences in peculiarities are, I consider, sufficient to justify me in believing that I have reproduced the solution of the celebrated Greek geometer. And I feel the better pleased at this as it clears up a long disputed point concerning the claims of the rival *'restorations'* given by Victa and Simson to the case of the problem in which one of the circles as C is infinitely small.

It will be seen that Vieta's solution, in the most improved form, is the same as that of Apollonius.

And here, before closing my notes on this celebrated problem, I may observe that Dr. Robert Simson, like many others, certainly misunderstood the object of propositions 116, 117 and 118 of Book VII. of Pappus' Mathematical Collections; and that through this he was led to imagine he restored or reproduced the proposition to which they were intended as subsidiary.

However, as Dr. Simson's remarks are interesting in a historical point, I will give them as translated from the Appendix to his *Opera Reliqua,* by Professor Davies. They are as follows:—

"In the Seventh Book of the Mathematical Collections of Pappus Alexandrinus (every admirer of the ancient geometrical analysis ought to rejoice that this work has been preserved to our times), among the lemmas which that most eminent writer has handed down, there exists a problem for one of the tangencies of Apollonius, namely, in Prob. 117, B. VII; in which it is required, when a circle being given by position and three points in a straight line, to inflect from two of the points two lines meeting in the circumference, so as to make the two points in which they intersect the circle and the third given point in the same straight line. It is not difficult to investigate the rest of the lemmas which are subsidiary to the problems on the tangencies; and some of these Vieta has used in his *Apollonius Gallus;* but to what problem the aforesaid lemma could be subsidiary, neither Vieta nor any other geometer has attempted to conjecture.

"Often, indeed, have I resolved the subject in my mind, but I have never succeeded in arriving at any satisfactory conclusion, except that the lemma, by no uncertain marks, appeared to be necessary for the following problem:—Two circles and a point being given by position, it is required to describe a third circle which shall touch the given circles and pass through the given point. In what manner, however, the lemma might be subsidiary to this problem I did by no means perceive. I have directed my attention to the solutions of Vieta and others, hoping that by chance I might hit upon the analysis requiring this lemma, but in vain; until this day, after various trials, I discovered the true analysis of Apollonius, to which, indeed, both this Prob. 117 of Pappus, as well as Props. 116 and 118 are manifestly subsidiary.—February 9, 1734."

How such an able geometer could look so long in vain for a solution to the tangencies which might implicate the 117th proposition of Pappus' 7th Book, I am at a loss to understand; though it evidently accounts for his implied opinion that the general problem of *the three circles* was originally referred to the particular case of *two circles and a point.*

Indeed, I may mention that the solution which I give as that of Apollonius was the first one which suggested itself to me for the general question of the three circles; though not exactly in the form in which I now present it: for after arriving at the point in the analysis in which M is shown to be found, I proceeded as follows:—

Since HI is parallel to PM, a circle through M, P and D touches the circle A in D; and ∴, since P and M are known, this touching circle MDP is known; and hence ODE, PDF, and circle DEF are known.

I may further note, that we may give another method of solution implicating Pappus' lemma, by supposing K the point in which DK parallel to OPQ cuts circle DEF, and V that in which FK cuts OPQ.

For as the angle FE right to K = DE right to K, it is= OE right to V and ∴ QV.QO being = QE.QF, the point V is known.

And if T be the point in which the tangent at D cuts OPQ; then since the angle FK right to D = DK right to T, it is= TP right to D, and ∴, PT.PV being = PD.PF, the point T is known. Hence the tangent TD to circle A is known, and ∴ ODE, PDF and circle DEF are known.

It is evident this method holds graphically good only when A is finite and C neither infinitely small, nor infinitely small in respect to the circle B. It applies to the case in which A is finite and B and C infinitely greats; but does not to that in which A is finite and B and C infinitely smalls.

Similar solutions to the two just indicated are obvious from the "*Involution*" *Theory* as unfolded in the Géométrie Supérieure:—

1. Let D, E, F be the points of contact with the given circles, and OPQ the known centres of similitude, in a straight line, through which DE, DF and EF pass. If G and H be the other points in which DE and DF cut the circle A, then GH is parallel to EF, and OG : OE : : rad A : rad B; and hence S the point in which GH cuts OQ is known.

Now, if we suppose T the point in which a tangent to the circle A at D cuts OQ; then, since we may regard GHDD as an inscribed quadrilateral in circle A having the side DD infinitely small, it follows that the straight line OPST is cut in "*involution*" by the circle A and the pairs of opposite sides of the quadrilateral; but the circle A is known, as also the points O, P and S; therefore the point T can be found as follows (see Chasles' *Géométrie Supérieure,* page 150):—Assume any point U on the circumference A, and describe the circle OPU; through U, S, and the other point V in which the circles A and OPU intersect, describe a circle UVS: then will this circle UVS give the point T in its other intersection with OPS. Hence we know the tangent TD to circle A, &c.

2. We can find the tangent NH in a similar manner. Thus:—Through W any point in the circumference A describe the circle PWS; through O, W, and the point X in which this circle cuts the circle A describe a circle: then will this circle give N in its other intersection with the line SPO. Hence the tangent NH, &c.

3. Or, since the required circle DEF evidently cuts OPQ in known points, real or imaginary, it is obvious that if through any point D′ we draw two straight lines OD′ and PD′, and on them take D′E′ and D′F′ such that OD′.OE′ = OD.OE, and PD′.PF′ = PD.PF, then will the circles D′E′F′ and DEF have OPQ as radical axis; and we can find the tangent DT as follows:—Describe the circle OPD′, and through Q, D′, and the other point K of intersection of the circles OPD′ and D′E′F′ describe a circle: then will the other point in which the circle D′KQ cuts OPQ be the required point T from which to draw the tangent TD to the circle A.

Montucla gives a very curious history of this problem. He says:—"Vieta, in a dispute with Adrian Romanus, proposed this problem. The solution given by Romanus, though obvious, was very indifferent, viz., by determining the centre of the required circle by the point of intersection of two hyperbolas. Vieta solved it very elegantly in his *Apollonius Gallus,* printed at Paris in 1600: his solution is the same as that given in *Newton's Universal Arithmetic.* Another solution may be seen in Lemma 16, Book I., of the Principia (this question being there necessary for some determinations in Physical Astronomy), where Newton, by a remarkable dexterity, reduced the two higher loci of Romanus to the intersection of two straight lines. Moreover, Descartes attempted to solve this problem by algebraic analysis, but without success; for, of the two solutions which he derived from thence, he himself acknowledges that one furnished him with so complicated an expression, that he would not undertake to construct it in a month; while the other, though somewhat less complicated, was not so very simple as to encourage him to set about the construction of it. Lastly, the Princess Elizabeth of Bohemia, who it is well known honored Descartes with her correspondence, deigned to communicate a solution to this philosopher; but, as it is deduced from the algebraic calculus, it labours under the same inconveniences as that of Descartes."

Euler, Fuss, T. Simpson, and other eminent analysts have given algebraic solutions, though not at all commensurate with the requirements of the problem. T. Simpson has also given a geometrical solution in the appendix to his Elements of Geometry, which, in reality, does not differ in principle from Newton's in the Principia.

These solutions, like those of Euler, are very imperfect, though complete ones of a similar nature, and much more simple, can be easily formed.

The late John Mulcahy, L.L.D., professor, Queen's College, Galway, after giving Gaulthier's solution as improved by Gergonne, in article 68 of his *Principles of Modern Geometry*, again returns to the subject in article 95, and deduces Gaulthier's original method depending on the circle which cuts the three given ones orthogonally: this of course labours under the disadvantage of being inapplicable when the radical centre is within the three given circles.

Those who are acquainted with the *Principles of Modern Geometry*, or the writings of the late Professor Davies, of the Royal Military Academy, will at once see that all my methods are applicable when, instead of *three circles in a plane*, there is given *three circles on the surface of a sphere*. The only difference being that straight lines, whether in data or solution, will be represented by great circles of the sphere.

My solutions have also analogous ones answering to the following celebrated problem, which was proposed by Descartes to Fermat:—"*Suppose four things*, A, B, C, D, *to be given in position, consisting of points, planes, and spheres, which may be taken of any one of these kinds exclusively, or of any two of the kinds, or of all of the three kinds; it is required to describe a sphere which shall pass through each of the given points, and touch the given planes or spheres.*"

All we have to do is (as in considering the Apollonian problem) to form the solution for the general case in which the data is four spheres, and then make the modifications necessary when we suppose one or any number of the spheres to become infinitely small or infinitely great, or—in other words—when we suppose one, two, three, or all of the spheres to be replaced by points or planes.

The analysis similar to my first solution is evidently as follows:—

ANALYSIS.

Let a, b, c, d be the respective points of contact of the required sphere with the given spheres A, B, C, D.

Then the straight lines ab, ac, ad, pass respectively through O, P, Q, known vertices of similitude to the pairs of spheres AB, AC, AD.

Now, if a' be any point in the surface A, and that b', c', d', are the respective points in the surfaces of B, C, D, made by the straight lines a'O, a'P, a'Q which are dissimilar to that of a' on the surface of A, it is evident $Oa'.Ob' = Oa.Ob$, $Pa'.Pc' = Pa.Pc$, and $Qa'.Qd' = Qa.Qd$, and therefore that the spheres $a'b'c'd'$, $abcd$, have the plane OPQ as radical plane.

This being borne in mind, it is evident that the tangent plane to any of the given spheres at the point of contact will cut the plane of section of this sphere and the sphere $a'b'c'd'$ in a straight line situated in the plane OPQ.

But as we may assume the point a' anywhere in the surface of A, we know the resulting points $b'c'd'$, and the sphere $a'b'c'd'$, and its planes of intersection with the spheres A, B, C, D, and also the intersections of these planes with plane OPQ, and the tangent planes from these lines to the given spheres, and ∴ the sought sphere of contact.

And it may be remarked that to each plane of similitude OPQ there are two answerable spheres $abcd$ whose centres (as also the centre of the corresponding sphere $a'b'c'd'$) are on the perpendicular from the radical centre of the four given spheres to the plane OPQ; and, moreover, that as there are eight planes of similitude OPQ there are sixteen answerable touching spheres (real or unreal in pairs). We may further remark that this solution furnishes a proof that the twelve vertices of similitude of the four given spheres lie in sixes in the eight planes OPQ, and are the vertical points of a complete octahedron.

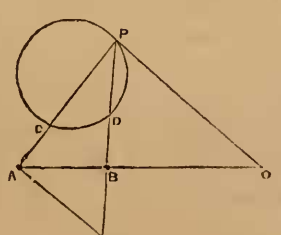
It is also easy to see that the chords aa of the sphere A pass *respectively* through the poles of the corresponding planes OPQ in respect to sphere A, and *all* through R the radical centre of the four given spheres; and therefore, it follows, that the eight lines in which the planes of intersection of the eight spheres $a'b'c'd'$ with the sphere A cut the corresponding eight planes OPQ of similitude, are situated in the *polar plane* of the radical centre R in respect to the sphere A, &c.

Or we might determine the point a (and hence b, c, d) from the following considerations:—Since Oa.Ob, and Pa.Pc are of known magnitudes and that aO.ab has to aP.ac a known ratio, therefore the point a must be on the surface of a known sphere having its centre in the straight line through O and P.

Similarly, since Pa.Pc and Qa.Qd are of known magnitudes and that aP.ac has to aQ.ad a known ratio, therefore the point a must be on the surface of a known sphere having its centre in the straight line through P and Q. Hence, as the point a is on the sphere A, it follows that it must be a point of intersection of the circular traces made on the sphere A by the two known spheres having their respective centres in the straight lines PO and PQ.

The other solutions to Descartes' *problem of the spheres*, which are analogous to those I have given to Apollonius' *problem of the circles*, may be easily made:—the tangent to two circles in the plane being represented by a plane touching three spheres, &c. And the actual operations are very simple when performed according to Monge's practical processes of the geometry of figured space, known by the name of "Descriptive Geometry."

APOLLONIAN LOCI PROBLEM.



Given two points A, B, *and the magnitudes of four lines* a, b, c, d, *to find the locus of a point* P, *such that* AP² + a.b :